

F-theory & M-theory perspectives on $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions.

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Abstract

Deformations of the original F-theory background are proposed. These lead to multiple new dualities and physical phenomena. We concentrate on one model where we let seven-branes wrap a multi-centered Taub-NUT space instead of \mathbb{R}^4 . This configuration provides a successful F-theory embedding of a class of recently proposed four-dimensional $\mathcal{N} = 2$ superconformal (SCFT) à la Gaiotto. Aspects of Argyres-Seiberg duality, of the new Gaiotto duality, as well as of the branes network of Benini-Benvenuti and Tachikawa are captured by our construction. The supergravity theory for the conformal case is also briefly discussed. Extending our construction to the non-conformal case, we find interesting cascading behavior in four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry. Since the analysis of this unexpected phenomenon is quite difficult in the language of type IIB/F-theory, we turn to the type IIA/M-theory description where the origin of the $\mathcal{N} = 2$ cascade is clarified. Using the T-dual type IIA brane language, we first start by studying the $\mathcal{N} = 1$ supersymmetric cascading gauge theory found in type IIB string theory on p regular and M fractional D3-branes at the tip of the conifold. We reproduce the supersymmetric vacuum structure of this theory. We also show that the IIA analog of the non-supersymmetric state found by Kachru, Pearson and Verlinde in the IIB description is metastable in string theory, but the barrier for tunneling to the supersymmetric vacuum goes to infinity in the field theory limit. We then use the techniques we have developed to analyze the $\mathcal{N} = 2$ supersymmetric gauge theory corresponding to regular and fractional D3-branes on a near-singular K3, and clarify the origin of the cascade in this theory.

Résumé

Différentes déformations de la géométrie originale de la théorie F sont proposées. Ces dernières génèrent une multitude de nouvelles dualités ainsi que de nouveaux phénomènes physiques. Nous nous concentrons sur un seul modèle où les membranes en sept dimensions spatiales s'enveloppent autour d'un espace Taub-NUT avec multicentres au lieu de l'espace \mathbb{R}^4 original. Cette configuration génère avec succès la réalisation, en théorie F, d'une famille de théories de jauge superconformes en quatre dimensions avec $\mathcal{N} = 2$ supersymétries nouvellement proposées par Gaiotto. Deplus, plusieurs aspects de la dualité d'Argyres-Seiberg, de la nouvelle dualité de Gaiotto ainsi que du réseau de membranes de Benini-Benvenuti et Tachikawa sont réalisés par notre construction. La théorie de supergravité pour le cas conforme est brièvement discutée. La généralisation de notre construction au cas non-conforme mène à l'observation surprenante de cascade chez les théories de jauge avec $\mathcal{N} = 2$ supersymétries en quatre dimensions. Puisque l'analyse de ce phénomène est difficile dans le langage de type IIB/ théorie F, nous nous tournons vers le type IIA/théorie M où l'origine de ce phénomène est élucidée. En utilisant le langage des membranes en type IIA sous la dualité-T, nous débutons par l'étude de cascade chez les théories de jauge avec $\mathcal{N} = 1$ supersymétrie tel que présenté en type IIB avec p membranes D3 régulières et M membranes D3 fractionnaires situées au bout d'un espace conifold. Nous reproduisons avec succès la structure du vide supersymétrique de cette théorie. Aussi, nous démontrons que l'analogue en type IIA des états non-supersymétriques découverts par Kachru, Pearson et Verlinde en type IIB sont métastables en théorie des cordes alors que la barrière permettant de passer au vide supersymétrique tant vers l'infini dans la limite de la théorie des champs. Nous utilisons finalement les techniques que nous avons développées afin d'analyser la théorie de jauge supersymétrique avec $\mathcal{N} = 2$ correspondante à des membranes D3 régulières et fractionnaires sur un espace K3 presque singulier et clarifions l'origine du mécanisme de cascade dans cette théorie.

DEDICATION

Bismillahir Rahmanir Rahim
Al Hamdu Lillahi Rabbil Alamin

This thesis is dedicated to my family for their unwavering faith and support.

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Preface

Statement of Originality

The results presented in this thesis constitute original work that was published in the following articles:

- **Chapter 3** K.Dasgupta, J. Seo and **A. Wissanji** (2012), “F-theory, Seiberg-Witten curves and $\mathcal{N} = 2$ Dualities,” Journal of High Energy Physics **1202**, 146, 117pp.
- **Chapter 4** D.Kutasov and **A. Wissanji** (2012), “IIA Perspective on Cascading Gauge Theory” arXiv: 1206.0747[hep-th], 43pp.

Chapter 3 is based on what was referred in [27] as model 2. We present a deformation of Sen’s original F-theory geometry which enabled us to embed in F-theory a class of Gaiotto new $\mathcal{N} = 2$ SCFT as well as several aspects of Argyres-Seiberg duality, Gaiotto duality, and the Benini-Benvenuti-Tachikawa brane network. We are therefore able to present a simple geometric brane picture which captures many intricacies of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. We also propose a type IIB/F-theory non-conformal construction which seems to have all the right ingredients to lead to a cascade mechanism in four-dimensional $\mathcal{N} = 2$ SYM theories. Chapter 4 is based on [56] where we study the $\mathcal{N} = 1$ cascade mechanism of Klebanov-Strassler and reproduce the supersymmetric vacuum structure of this theory using type IIA/M-theory brane constructions. We show that the type IIA analog of the non-supersymmetric state of Kachru-Pearson-Verlinde is metastable in string theory but the barrier for tunnelling to the supersymmetric vacuum goes to infinity in the field theory limit. We finally analyze the $\mathcal{N} = 2$ supersymmetric gauge theory using type IIA/M-theory and clarified the origin of the cascade in this theory.

Contribution of the author

[27] was work done in collaboration with Professor Keshav Dasgupta and Jihye Seo from McGill University. I proposed the original idea that led to model 2 in [27]. This was the stepping stone which led to further studies and generalizations in the paper. In this article, I participated in detailed discussions and calculations at every step of the analysis, leading to many results that were included in the article. Finally, I wrote various appendices in the paper.

[56] was work done in collaboration with Professor David Kutasov from the Enrico Fermi Institute at the University of Chicago. In this paper, I proposed to study the $\mathcal{N} = 2$ cascade, wrote the original draft of the associated section in the article, and contributed to detailed discussion at every step of the analysis which led to many results that were included in the article.

Table of Contents

Abstract	ii
Résumé	iii
DEDICATION	iv
ACKNOWLEDGEMENTS	v
Preface	vii
List of Figures	xi
List of Tables	xv
1 Introduction	1
2 Aspects of String Theory and Branes	5
2.1 String theory parameters	5
2.2 Strings in String Theory	6
2.2.1 Open or closed strings	6
2.2.2 Chan-Paton factor	8
2.3 Type IIA and Type IIB	11
2.3.1 S-duality	14
2.4 D-branes, Orientifolds and NS5-branes	16
2.4.1 Dirichlet-branes	16
2.4.2 Nonabelian gauge groups from stacks of Dp-branes	18
2.4.3 Orbifolds and Orientifolds	19
2.4.4 NS5-branes	20
2.4.5 Webs of branes	22
2.5 M-theory	25
2.6 Seiberg-Witten theory	27
2.7 Argyres-Seiberg duality	34
2.8 F-theory	37

2.9	F-theory embedding of $\mathcal{N} = 2$ SUSY $d = 4$	40
2.10	Possible background deformations	45
3	F-theory embedding of new supersymmetric gauge theories . . .	49
3.1	Introduction	49
3.2	Multiple $D3$ -branes probing seven-branes on a Taub-NUT background	50
3.3	Brane anti-brane on a Taub-NUT background	52
3.4	Anomaly inflow, anti-GSO projection and brane transmutation .	56
3.5	Mapping to Gaiotto theories and beyond	62
3.5.1	Mapping to the type IIB brane network models	62
3.5.2	The UV/IR picture and gravity duals	65
3.5.3	Mapping to the conformal cases	74
3.5.4	Beyond the conformal cases	79
4	Type IIA perspective on cascading gauge theories	84
4.1	Introduction	84
4.2	Classical theory	87
4.3	Quantum theory	99
4.3.1	$p = 0$	99
4.3.2	$0 < p < M$	102
4.3.3	$p = M$	104
4.3.4	$p > M$	107
4.4	Non-supersymmetric brane configurations	111
4.4.1	Non-supersymmetric vacua with $\xi \neq 0$	111
4.4.2	Adding \bar{D} -branes to KS	114
4.5	$\mathcal{N} = 2$ cascade	119
4.5.1	Classical moduli space	120
4.5.2	Quantum moduli space	121
5	Conclusion	126
A	Aspects of the IIA description of $\mathcal{N} = 2$ SQCD	129
	References	135

List of Figures

Figure		Page
2.1	Fundamental domain in τ for $SL(2, \mathbb{Z})$ (blue) and in $\tilde{\tau} \equiv 2\tau$ for $\Gamma^0(2)$ (red). Edges of the domains are identified under reflection by $\text{Re}\tau = 0$. Figure extracted from [7] and modified.	35
2.2	In F theory a single D3-brane probing an orientifold seven-plane background maps to the classical picture of $Sp(2)$ Seiberg-Witten theory.	41
2.3	The quantum corrections in the $Sp(2)$ theory maps to the splitting of the orientifold plane into two (p, q) seven-branes in F-theory.	44
3.1	Multiple $D3$ -branes probing seven-branes on a multi Taub-NUT geometry.	51
3.2	In fig (a) the singularities of Taub-NUT space are shown. The compact direction is the Taub-NUT circle that is fibered over the base. The various points at which the circle (which is along x^6 direction in the text) degenerate are the singular points. Between two singular points form a two-cycle, as shown in fig (b), once we assume that the x^6 circle is degenerating along a line parametrized by the x^7 coordinate. Thus the P^1 's are labelled by $x^{6,7}$ coordinates in the text.	52
3.3	The gauge fluxes on the D5 and the $\overline{D5}$ branes are changed with time. The red and the black lines denote two different changes with the same initial and final values.	59
3.4	HW brane creation: solid vertical lines are NS5-branes (23456), dashed vertical line is D5-brane (56789), solid horizontal line is a $D3$ -brane along (156)	60

3.5	HW brane creation: solid vertical lines are NS5-branes (023456), dashed vertical line is D5-brane (056789), solid horizontal line is a $D3$ -brane along (0156), horizontal dashed line is an anti- $D3$ -brane (0156)	61
3.6	The simplest brane junction from our Taub-NUT configuration.	63
3.7	Under special arrangement of the seven-branes, delocalization and T-dualities map our model to the brane network studied in [14]. For other configurations there are no simple map to the brane networks. The blue patches on both sides represent the seven-branes. The r $D5-\overline{D5}$ pairs are wrapped on vanishing 2-cycles of a multi Taub-NUT space with m centers.	66
3.8	$D5-\overline{D5}$ branes wrapped on 2-cycles of a multi Taub-NUT space. When each of these N set of $D5-\overline{D5}$ pairs wrap vanishing cycles of the multi Taub-NUT space, they can give rise to $[SU(N)]^m$ gauge groups where m denote the number of vanishing Taub-NUT 2-cycles. In the next figure various such pairs are broken and moved along the Coulomb branch.	67
3.9	The brane junction that we get in the left of the figure may further be truncated by breaking the NS5 and the (p, q) five-brane and moving along the Higgs branch. However as we argue below, this may not be required to see the underlying dualities.	75
3.10	On the left is a non-susy configuration that appears from naive T-duality of the Taub-NUT model, as the branch cuts of the seven-brane (shown as solid black circles) would modify the parallel NS5-branes configuration from the local axion-dilaton charges. This is similar to the deformation of a string-network from background axion as shown in [23]. The arrangement of the seven-branes along x^6 direction come from the original framework of the wrapped seven-branes with A_6 switched on. T-duality convert A_6 to x^6 shown on the left. On the right is the susy configuration by moving the seven-branes across the “bent” NS5-branes.	76

3.11	A simple non-conformal deformation of the theory where the multiple $D5-\overline{D5}$ branes are probing a $E_7 \times E_8$ singularity. The global symmetry will have additional $U(1)$ that are not shown in the figure. In the language of [34] the filled circles are A -branes, the empty circles are B -branes and the filled squares are C -branes.	81
4.1	The IIA brane configuration that realizes the cascading gauge theory. The $NS5$ -branes are depicted in green, and are connected by $M + p$ $D4$ -branes on one side of the x^6 circle (whose radius is R_6) and by p $D4$ -branes on the other	88
4.2	A generic point in the classical moduli space. p $D4$ -branes wrap the x^6 circle and can move in the transverse space; M are stretched between the fivebranes and give rise to $SU(M)$ $\mathcal{N} = 1$ SYM	91
4.3	Turning on an FI term in general leads to branes at an angle and breaks SUSY.	93
4.4	The baryonic vacuum for $\xi \neq 0$ corresponds to a stack of M $D4$ -branes connecting the fivebranes and winding k times around the circle ($k = 2$ in the figure).	94
4.5	Open string tachyon condensation connects vacua with different values of l	96
4.6	The ground state of the brane system with non-zero ξ , viewed in the covering space of the x^6 circle. $M - \tilde{p}$ fourbranes have winding k , while \tilde{p} have winding $k + 1$. The specific case exhibited is $k = \tilde{p} = 1$, $M = 2$, $p = 3$	97
4.7	The quantum ground state of the brane system with $p = 0$ is described by the curved fivebrane (4.35), (4.36).	101
4.8	The quantum moduli space of the brane system with $M > p > 0$ is described by p $D4$ -branes wrapping the x^6 circle in the vicinity of the curved fivebrane of figure 4.7.	103
4.9	The baryonic vacua of the brane system with $p = M$, viewed in the covering space of the x^6 circle. Vertical dashed lines are separated by $2\pi R_6$ and are identified on the circle. . . .	105
4.10	The baryonic branch of the brane system with $p = kM$, with \bar{p} $\overline{D4}$ -branes wrapping the circle ($k = 1$ in the figure).	115

4.11	The configuration of figure 4.10 is unstable to decay to that depicted here.	116
4.12	The low energy description of the metastable vacuum of figure 4.11 consists (classically) of $M - \bar{p}$ $D4$ -branes (red) and \bar{p} $\bar{D}4$ -branes (blue) stretched between the $NS5$ -branes.	117
4.13	The IIA brane configuration that realizes the $\mathcal{N} = 2$ supersymmetric cascading gauge theory.	119
4.14	The brane description of \mathcal{M}_n , a component of the classical moduli space, has n fourbranes wrapping the circle moving in the transverse \mathbb{R}^5 (in general away from the fivebranes), and $M + p - n$ resp. $p - n$ fourbranes connecting the fivebranes and distributed in the v plane.	120
A.1	The brane description of $\mathcal{N} = 2$ SQCD with N_c colors and N_f flavors.	129
A.2	The brane description of $\mathcal{N} = 2$ SQCD at finite ξ	130
A.3	The origin of the baryonic branch of $\mathcal{N} = 2$ SQCD in the quantum theory. The dashed line corresponds to the UV cutoff $ v = v_\infty $	131
A.4	The classical limit of the small v limit of the brane configuration of figure A.3.	132

List of Tables

Table		Page
2.1	IIA M-theory correspondence. Directions along which the branes are stretched is indicated.	27
2.2	Possible background deformations of the F-theory geometry .	46
3.1	Naive link between our model in the IR and type IIA existing construction	51
3.2	The orientations of various branes and fluxes in our set-up. The dashed lines for the branes are the directions parallel to the world-volume of the branes; and for the fluxes and Taub-NUT space are the directions along which we have non-trivial fluxes and metric respectively. The stars denote orthogonal spaces. Supersymmetry is preserved in the presence of vanishing cycles and background fluxes.	56
3.3	Variation of world-volume fluxes with time	59
3.4	The orientations of various branes in our set-up. Same as the earlier table but now the flux informations are not shown. .	63

Chapter 1

Introduction

The primary goal of this thesis is to show that many new facets of non-abelian gauge theories with $\mathcal{N} = 2$ supersymmetry (susy) in four dimensions can be revealed by using the language of branes in string theory. In their proper regime of validity, branes capture all the physics contained in supersymmetric field theories and provide insights on new physical phenomena by generating geometric pictures of the intricacies of these field theories. Moreover, they are powerful tools for understanding certain dualities occurring in supersymmetric field theories. In particular, branes in string theory shed new light on the strongly coupled regime of supersymmetric non-abelian gauge theories both in the conformal and non-conformal cases. The long term goal of this research direction is that it might lead to a better understanding of some aspects of physical phenomena occurring at strong coupling in non-supersymmetric non-abelian gauge theories and about which little is currently known. We will however not address this question in this thesis.

Non-abelian, non-supersymmetric gauge theories such as Quantum Chromodynamics (QCD) are the foundation on which our understanding of the dynamics of elementary particles in the Standard Model lies. Given the importance of such theories, it is surprising to realize that there is still much to learn about them. For instance, these theories are often asymptotically free meaning that they are free in the

ultra-violet energy (short distance). These theories are also strongly coupled at infra-red energy (long distance). In the latter regime, all our perturbative (weakly coupled) field theory techniques fail. Trying to study these strongly coupled non-abelian gauge theories is one of the biggest challenge of modern theoretical high energy physics.

Seiberg and Witten provided in [68, 69] a much celebrated breakthrough in understanding the strongly coupled regime of a certain family of gauge theories namely Supersymmetric Yang-Mills (SYM) theories. Their work was achieved through better understanding of the implications of supersymmetry and use of a known duality which allowed them to probe the strong coupling regime of supersymmetric field theories by providing a dual weakly coupled picture; inverting the electric matter for the magnetic one in the process. The work of Seiberg and Witten led to exact results on the vacuum structure of non-abelian supersymmetric gauge theories, both with and without matter content. These results became the stepping stone for many generalization to higher rank gauge groups, new superconformal field theories, and more complicated dualities. In addition to the plethora of new mathematical applications they provided, these supersymmetric field theories became toy models for studying theories such as QCD since they capture some phenomena which also occurs in non-supersymmetric non-abelian gauge theories.

In recent years, it was shown that branes - extended object in string theory [66]- are powerful objects that provide geometric and tractable descriptions of supersymmetric gauge theories. From the point of view of theories living on branes, gauge theories appear as effective low energy descriptions which are valid in prescribed regions of the moduli space of vacua. Different brane pictures have different descriptions depending on which region of the moduli space of vacua one is interested in studying; sometimes providing insights into regions which don't even have field theoretic descriptions. In addition to shedding light on relations between such field theories, we will see that the simple nature of branes allows one to unveil new physics hidden in the language of field theory. The brane description that we will mostly be concern with throughout this thesis is that of IIB and F-theory as proposed by Vafa in [76] as well as that of type IIA and M-theory put forward by Witten [79].

In aiming to understand the interplay between the brane language and supersymmetric gauge theories, we will review in Chapter 2 what we believed to be the starting point of this research direction, namely the embedding of Seiberg-Witten theory in F-theory by Sen [72] and Banks, Douglas and Seiberg [11]. We will then describe possible deformations of the original F-theory background which enabled us to not only describe recently proposed field theories and their associated dualities but also discover new physics using the language of branes. In Chapter 3, we will concentrate on one model where we let seven-branes wrap on a multi-centered Taub-NUT space instead of \mathbb{R}^4 . This configuration provides a successful embedding in F-theory of a class of recently proposed four-dimensional $\mathcal{N} = 2$ SCFT à la Gaiotto [35]. Aspects of Argyres-Seiberg duality [7], of the new Gaiotto duality [35], as well as of the brane network of Benini-Benvenuti and Tachikawa [14] will be captured by our construction. The supergravity theory for the conformal case will also be briefly discussed. Extending our construction to the non-conformal case, we will find interesting cascading behavior in theories with $\mathcal{N} = 2$ supersymmetry in four dimensions [67, 15]. Since the analysis of this unexpected phenomenon will be limited by the difficulties of the type IIB/F-theory language, we will turn, in Chapter 4, to type IIA/M-theory where the origin of $\mathcal{N} = 2$ cascade mechanism will become clear. Using the T-dual type IIA brane language, we will first start by studying the $\mathcal{N} = 1$ supersymmetric cascading gauge theory found in type IIB string theory on p regular and M fractional D3-branes at the tip of the conifold [54]. We will reproduce the supersymmetric vacuum structure of this theory [30]. We will then show that the IIA analog of the non-supersymmetric state found by Kachru, Pearson and Verlinde [52] in the IIB description is metastable in string theory, but the barrier for tunneling to the supersymmetric vacuum goes to infinity in the field theory limit. We will then use the techniques we will have developed to analyze the $\mathcal{N} = 2$ supersymmetric gauge theory corresponding to regular and fractional D3-branes on a near-singular K3, and clarify the origin of the cascade in this theory. We will end with Chapter 5 where a discussion of possible extensions of this work will be presented. An appendix to Chapter 4 is also included in this thesis.

We start Chapter 2 with a review of some basic notions of string theory and branes that will be useful throughout the thesis. The notation used in the thesis is as follows: $1 + 9$ spacetime dimensions in string theory are labeled by (x^0, x^1, \dots, x^9) . The eleventh spatial dimension of M-theory is x^{10} . The corresponding Dirac matrices are denoted by Γ^μ , $\mu = 0, 1, \dots, 9$ with algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{1.1}$$

and the metric has the signature $(- + \dots +)$.

Chapter 2

Aspects of String Theory and Branes

In this section, we will review some useful properties of type IIA and type IIB String Theory as well as their embedding in M-theory and F-theory respectively. This discussion will be accompanied by a description of the branes in each regime.

2.1 String theory parameters

The protagonists of this story are strings. These are one spatial dimensional objects with characteristic length scale denoted by l_s . The string length scale l_s is related to the string tension T and to the open-string Regge slope parameter α' by the relations

$$T = \frac{1}{(2\pi\alpha')}; \quad \alpha' = \frac{1}{2}l_s^2. \quad (2.1)$$

Fundamental constants such as the speed of light c , Planck's constant \hbar and Newton's gravitational constants G form the Planck length l_p and the Planck mass m_p :

$$l_p = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6 \times 10^{-33} \text{cm}, \quad (2.2)$$

$$m_p = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19} \text{GeV}/c^2, \quad (2.3)$$

where $1 \text{ GeV} \approx (1 \times 10^9) \times (1.6 \times 10^{-19})$ joules. The UV cutoff of the theory is given by $1/l_s$ and the relation between l_s and l_p is [12]:

$$l_p = g_s^{1/3} l_s. \quad (2.4)$$

At energies far below the Planck energy E_p ($E_p = m_p c^2$), distances of the order of the Planck length can not be resolved and strings can be accurately approximated by point particles, like its the case in quantum field theory [12].

As it moves, the string sweeps out a two-dimensional surface in spacetimes called the string world sheet of the string. We will see later that a one spatial dimensional string can be generalized to a p spatial dimensional object denoted p -brane (the fundamental string having $p = 1$). The latter has tension T_p and sweeps out a $p + 1$ dimensional volume V in spacetime. The action of such p -brane is given by $S_p = -T_p V$ [12].

2.2 Strings in String Theory

The string sigma model action classically represents the world sheet action. The former is given by:

$$S_\sigma = \frac{T}{2} \int \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu d\sigma d\tau, \quad (2.5)$$

where α, β are world sheet indices and μ, ν are target space indices. $h_{\alpha\beta}(\sigma, \tau)$ is the world sheet metric, $h = \det h_{\alpha\beta}$. Throughout the text, we will encounter the function $X^\mu(\sigma, \tau)$ which describes, in the string sigma model action, the spacetime embedding of the string world sheet. The latter is parameterized by τ and σ where τ is the world sheet time coordinate and σ is the spatial coordinate, parametrizing the string at a given time [64].

2.2.1 Open or closed strings

Strings have different boundary conditions [12, 64] depending on whether they are opened or closed. A closed string is topologically a circle whereas an open string is topologically a line element. We let $0 \leq \sigma \leq \pi$. The two types of boundary

conditions which respect D-dimensional Poincaré invariance are stated below. They stipulate that no momentum is flowing through the ends of the string for all values of μ .

- The spatial coordinate σ is periodic for closed strings, leading to the condition

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau). \quad (2.6)$$

The endpoints are thus joined to form a loop and there is no boundary.

- Open string with Neumann boundary condition have a vanishing momentum with component normal to the boundary of the world sheet

$$\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, \pi) = 0. \quad (2.7)$$

The end of the open strings move freely in spacetime.

Throughout the text, we will need to consider a different sort of boundary condition which breaks Poincaré invariance:

- The Dirichlet boundary condition for open strings

$$X^\mu|_{\sigma=0} = X_0^\mu, \quad (2.8)$$

$$X^\mu|_{\sigma=\pi} = X_\pi^\mu, \quad (2.9)$$

which means that the two ends of the open string are fixed i.e $\delta X^\mu = 0$. X_0^μ and X_π^μ are constant with $\mu = 1, \dots, D-p-1$ where D is the dimension of spacetime and where Neumann boundary conditions are satisfied for the remaining $p+1$ coordinate.

X_0^μ and X_π^μ represent the positions of Dp -branes e.g p spatial dimensional Dirichlet branes. The defining property of the latter is that fundamental strings can end on them: the coordinates of the attached string satisfy Dirichlet boundary condition in the direction normal to the brane and Neumann boundary condition in the direction parallel to the brane. Dp -branes break Poincaré invariance unless $p = D-1$ [12, 64]. We will elaborate more on Dirichlet branes in the next few sections.

2.2.2 Chan-Paton factor

Under T-duality ($R \rightarrow 1/R$), an open string with Neumann boundary conditions becomes a open string with Dirichlet boundary conditions whose end points ends on Dp -branes. When one open string is in the presence of a stack of N Dp -branes, the open string carries at its endpoints quantum numbers (initially though of as quarks and antiquarks) transforming respectively in the N -dimensional representations \mathbf{R} , $\bar{\mathbf{R}}$ of a gauge group G . These N -valued labels, referred to as Chan-Paton charges, associate N degrees of freedom to each endpoints of the open string. Oriented open strings are characterized by complex representations \mathbf{R} for which $\mathbf{R} \neq \bar{\mathbf{R}}$ and thus have distinguishable endpoints. Unoriented open strings, on the other hand, have real representations \mathbf{R} which leads to $\mathbf{R} = \bar{\mathbf{R}}$ and thus have indistinguishable endpoints [46, 12].

Accordingly, for an oriented open string, one described the gauge group $U(N)$ by letting the charges located at $\sigma = 0$ transform under the fundamental representation \mathbf{N} while the degrees of freedom at $\sigma = \pi$ transform under the antifundamental representation $\bar{\mathbf{N}}$ of the gauge group. Unoriented open strings lead to orthogonal or symplectic groups with real fundamental representations at both $\sigma = 0$ and $\sigma = \pi$. This can be understood as follows. For strings with Dirichlet boundary conditions $\partial_\sigma X^i = 0$ at $\sigma = \{0, \pi\}$ for $\mu = i$, the mode expansion for $X^i(\sigma, \tau)$ is given by :

$$X^i(\sigma, \tau) = x^i + p^i \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^i e^{-in\tau} \cos n\sigma, \quad (2.10)$$

where σ goes from 0 to π along the string. Under orientation reversal, one interchanges the two ends of the string and let the parametrization of σ run in the opposite direction. We obtain the following mode expansion:

$$X^i(\pi - \sigma, \tau) = x^i + p^i \tau + i \sum_{n \neq 0} \frac{1}{n} (-1)^n \alpha_n^i e^{-in\tau} \cos n\sigma. \quad (2.11)$$

The coordinate transformation $\sigma \rightarrow \pi - \sigma$ and $\tau \rightarrow \tau$ used above is generated by the world sheet parity operator Ω whose properties are that $\Omega^2 = 1$ and its eigenvalues are

given by $\Omega = \pm 1$. As we see from the equations above, Ω exchanges α_n^i for $(-1)^n \alpha_n^i$ where α_n^i is a transverse oscillator

$$\Omega \alpha_n^i \Omega^{-1} = (-1)^n \alpha_n^i. \quad (2.12)$$

When the representations \mathbf{R} and $\bar{\mathbf{R}}$ are the same at both ends of the string, it is sensible to think that the quantum wave function of the string $X^i(\tau, \sigma)$ is invariant under the above orientation reversal operation. This leads to

$$|\Lambda(\alpha_n^i); b, a\rangle = \epsilon |\Lambda((-1)^n \alpha_n^i); a, b\rangle, \quad (2.13)$$

where $|\Lambda\rangle$ parametrizes the string state in the oscillator Hilbert space and a, b are additional labels carried by the state: parametrizing the Chan-Paton charges at the endpoints of the strings (transforming in the same representation for the case of unoriented strings). In the equation above, $\epsilon = \pm 1$. Generically, $|\Lambda; a, \bar{b}\rangle$ describes massless vector particles where the quantum number $a\bar{b}$ transform in the adjoint representation of the gauge group since massless vectors in consistent interacting theories always transform in the adjoint representation of the gauge group. Strings satisfying (2.13) are called unoriented open strings (with $b = \bar{b}$). What differentiates the orthogonal from the symplectic group for unoriented strings is whether their adjoint representation forms symmetric or antisymmetric states. In particular, for $SO(N)$ gauge group with $\mathbf{R}, \bar{\mathbf{R}}$ both equal to the N -dimensional real fundamental representation, the adjoint representation are given by antisymmetric matrices (antisymmetric part of the $\mathbf{R} \times \mathbf{R}$ representation) and the associated massless vector satisfies (2.13) with $(-1)^N = +1$ and $\epsilon = -1$ (for superstring). For $Sp(N)$ gauge group where $\mathbf{R}, \bar{\mathbf{R}}$ are the fundamental representation of the gauge group, the adjoint representation is the symmetric part of the $\mathbf{R} \times \mathbf{R}$ representation. The massless vector are those of (2.13) with $\epsilon = +1$ (for superstring). Recall that symplectic matrices are even-dimensional, leading to $Sp(N)$ with even N . To make contact with what we have already seen: for the oriented string case a and \bar{b} run over the fundamental \mathbf{N} and antifundamental

$\bar{\mathbf{N}}$ representation of $U(N)$ respectively where the adjoint representation is given by $\mathbf{N} \times \bar{\mathbf{N}}$ [46]. We focus on one oriented string in the presence of a stack of Dp -branes. Every state in the open-string spectrum has N^2 multiplicity, with N^2 massless vector states describing the $U(N)$ gauge fields. The basis of the open-string can be labelled as follows:

$$|\phi, k, ij\rangle, \quad (2.14)$$

where ϕ is the Fock space state, k the momentum and $i, j = 1, \dots, N$ label the Chan-Paton factors. As explained in [12], this state transform with charge $+1$ under $U(1)_i$ and charge -1 under $U(1)_j$. Arbitrary string states are described by a linear combination

$$|\phi, k, \lambda\rangle = \sum_{i,j=1}^N |\phi, k, ij\rangle \lambda_{ij}, \quad (2.15)$$

where there are N^2 hermitian matrices λ_{ij} called Chan-Paton matrices corresponding to the representation matrices of the $U(N)$ algebra. The string states then become matrices transforming in the adjoint representation of $U(N)$. The N^2 degrees of freedom of the oriented open string are not visible unless one puts that string on a stack of coinciding parallel Dp -branes and let the Chan-Paton factor at the endpoints of the string be $i = 1, j = N$ or vice versa. If the branes are separated, then there are N different massless $U(1)$ vectors and the resulting gauge theory is $U(1)^N$ and the Chan-Paton indices running from $i, j = 1, \dots, N$ correspond to the endpoints of different open strings, ending respectively on the i^{th} and j^{th} branes. This way of generating non-abelian gauge theories from the point of view of open oriented string with Chan-Paton factor ending on Dp -branes will be reviewed in the language of Dp -branes in a latter section.

2.3 Type IIA and Type IIB

Type IIA string theory is a non-chiral theory as it has $(1, 1)$ spacetime supersymmetry where the spacetime supercharges generated by left and right moving degrees of freedom Q_L , Q_R have opposite chirality [40]:

$$\Gamma^0 \cdots \Gamma^9 Q_L = +Q_L, \quad (2.16)$$

$$\Gamma^0 \cdots \Gamma^9 Q_R = -Q_R. \quad (2.17)$$

Type IIB on the other hand is a chiral theory since it has $(2, 0)$ spacetime supersymmetry, where the left and right moving supercharges have the same chirality [40]:

$$\Gamma^0 \cdots \Gamma^9 Q_L = Q_L, \quad (2.18)$$

$$\Gamma^0 \cdots \Gamma^9 Q_R = Q_R. \quad (2.19)$$

Ten dimensional type IIA supergravity theories can be obtained by dimensional reduction of a unique eleven dimensional supergravity theory which arises as the low energy limit of M-theory. We review here the field content of these theories.

Eleven dimensional supergravity includes the following bosonic fields: a metric G_{MN} and an antisymmetric three-form potential $A_{MNP} \equiv A_3$ with field strength F_4 . Here $M, N, P = 0, \dots, 10$. The fermionic content is given by the gravitino ψ_α^M with $\alpha = 1, \dots, 32$. The bosonic part of the action of eleven dimensional supergravity theory is given by [65]:

$$2\kappa_{11}^2 S_{11} = \int d^{11}x (-G)^{1/2} \left(R - \frac{1}{2} |F_4|^2 - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4 \right). \quad (2.20)$$

Dimensionally reducing eleven dimensional supergravity along a circle,

$$ds^2 = G_{MN}^{11}(x^\mu) dx^M dx^N \quad (2.21)$$

$$= G_{\mu\nu}^{10}(x^\mu) dx^\mu dx^\nu + \exp(2\sigma(x^\mu)) [dx^{10} + A_\nu(x^\mu) dx^\nu]^2, \quad (2.22)$$

one obtains type IIA supergravity. By the above process, one obtains from G_{MN} the following fields in type IIA:

- metric $G_{\mu\nu}$
- gauge field $A_\mu = G_{\mu,10}$
- scalar $\Phi = G_{10,10}$

where $\mu, \nu, \lambda = 0, \dots, 9$. On the other hand, the antisymmetric tensor A_{MNP} of eleventh dimensional supergravity gives rise to the following antisymmetric tensors in type IIA:

- $A_{\mu\nu\lambda}$
- $B_{\mu\nu} = A_{\mu\nu,10}$

After appropriate reparametrizations, the type IIA metric is given by [65]:

$$S_{IIA} = S_{NS} + S_R + S_{CS} \quad (2.23)$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) \quad (2.24)$$

$$S_R = -\frac{1}{4\kappa_{10}^2} \int d^{10}x (-G)^{1/2} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) \quad (2.25)$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4, \quad (2.26)$$

where $\tilde{F}_4 = dA_3 - A_1 \wedge F_3$. Neveu-Schwarz (NS) sector fields are $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ . The field strength associated to the potential B_2 is denoted by H_3 . The Ramond-Ramond (RR) sector fields are the gauge fields A_μ and $A_{\mu\nu\lambda}$. Their potentials and field strengths are respectively denoted by C_p and F_{p+1} . The vacuum expectation value of the exponential of the dilation Φ gives the coupling constant g_s of string theory. Consider the chain below where everything to the left of $(*F)$ electrically sources the p -brane denoted here by D_p while $(*F)$ and what is on its right magnetically sources the p -brane. $(*F)$ represents the Hodge dual of F , mapping a k -vector to an $(n-k)$ -vector where $n = 10$ here. Note that although the notation used below refers to RR sector fields, the relations between which fields sources which branes still holds for the NS sector.

$$D_p \rightarrow C_{p+1} \rightarrow F_{p+2} \rightarrow (*F)_{10-(p+2)} \rightarrow C_{10-(p+2)-1} \rightarrow D_{10-(p+2)-2}. \quad (2.27)$$

Here C_{p+1} denotes the potential and F_{p+2} is the field strength associated to the p -brane. We refer to branes that couple to the NS sector gauge field as NS-branes. On the other hand, branes charged under RR sectors fields are referred to as Ramond branes or D-branes. As an example, the chain above makes it clear that $B_{\mu\nu} \equiv B_2$ electrically sources a 1-brane (a fundamental string) and magnetically couples to a fivebrane (NS_5) through a six-form gauge field dual to $B_{\mu\nu}$. Since $B_{\mu\nu}$ is an NS sector field, we refer to these branes as NS-branes. Similarly, $D0$ -branes (point particles) are electrically charged under A_μ while the latter gauge field couples magnetically to $D6$ -branes. $D2$ and $D4$ -branes are electrically and magnetically sourced by the antisymmetric tensor field $A_{\mu\nu\lambda}$.

Type IIB supergravity is a ten-dimensional parity-violating theory. Its massless spectrum contains the same NS sector as in type IIA supergravity, namely $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ which couple to the corresponding NS string and fivebranes. The RR sector fields of type IIB is different than that of type IIA as it contains an additional scalar (0-form potential) called the axion C_0 which combines with the dilaton Φ to generate the complex coupling of type IIB:

$$\tau = C_0 + ie^{-\Phi}. \quad (2.28)$$

The antisymmetric tensors in the RR sector of type IIB are $\tilde{B}_{\mu\nu}$ and $A_{\mu\nu\lambda\rho}$. $\tilde{B}_{\mu\nu}$ couples electrically to D-string and magnetically to D5-branes. $A_{\mu\nu\lambda\rho}$ sources D3-branes both electrically and magnetically since this four-form is self dual: $*dA = dA$. This latter fact also implies the existence of a self-dual five-form field strength $*F_5 = F_5$ leading to $|F_5|^2 = 0$. The action of type IIB is given by [65]:

$$S_{IIB} = S_{NS} + S_R + S_{CS} \quad (2.29)$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) \quad (2.30)$$

$$S_R = -\frac{1}{4\kappa_{10}^2} \int d^{10}x (-G)^{1/2} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) \quad (2.31)$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3, \quad (2.32)$$

where

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3, \quad (2.33)$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \quad (2.34)$$

Although the condition $*\tilde{F}_5 = \tilde{F}_5$ can not be imposed on the action (2.29) or else the wrong equations of motion result, the field equations of (2.29) are consistent with the aforementioned condition- even if they don't imply it. The self duality of \tilde{F}_5 can therefore be added by hand on the solutions of the equations of motion as an additional constraint [65].

2.3.1 S-duality

An interesting fact about the low energy type IIB supergravity action (2.29) is that it can be written in an way that is invariant under $SL(2, \mathbb{R})$ symmetry [65]. To see this, consider the following coordinates:

$$G_{E\mu\nu} = e^{-\Phi/2} G_{\mu\nu} \quad (2.35)$$

$$\tau = C_0 + ie^{-\Phi} \quad (2.36)$$

$$M_{ij} = \frac{1}{\text{Im}\tau} \begin{bmatrix} |\tau|^2 & -\text{Re}\tau \\ -\text{Re}\tau & 1 \end{bmatrix} \quad (2.37)$$

$$F_3^i = \begin{bmatrix} H_3 \\ F_3 \end{bmatrix}. \quad (2.38)$$

The metric (2.29) can then be rewritten in the following $SL(2, \mathbb{R})$ invariant way:

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G_E)^{1/2} \left(R_E - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{2(\text{Im}\tau)^2} - \frac{M_{ij}}{2} F_3^i \cdot F_3^j - \frac{1}{2} |\tilde{F}_5|^2 \right) \\ &- \frac{\epsilon_{ij}}{4\kappa_{10}^2} \int C_4 \wedge F_3^i \wedge F_3^j, \end{aligned} \quad (2.39)$$

where G_E (2.35) is the Einstein metric and where the coupling and fields transform as:

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad (2.40)$$

$$F_3^{i'} = \Lambda_j^i F_3^j \quad \Lambda_j^i = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \quad (2.41)$$

$$\tilde{F}'_5 = \tilde{F}_5 \quad G'_{E\mu\nu} = G_{E\mu\nu}, \quad (2.42)$$

with $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$. Although the low energy effective action of IIB supergravity has a global $SL(2, \mathbb{R})$ symmetry, τ is invariant under an $SO(2, \mathbb{R})$ subgroup, so the moduli space is locally the coset space $SL(2, \mathbb{R})/SO(2, \mathbb{R})$. It is in fact known that the full type IIB superstring theory (considering quantum effects) has the discrete subgroup $SL(2, \mathbb{Z})$ as an exact symmetry of the theory. The latter transforms the fields in (2.29) as:

$$\Phi' = -\Phi \quad G'_{\mu\nu} = e^{-\Phi} G_{\mu\nu} \quad (2.43)$$

$$B'_2 = C_2 \quad C'_2 = -B_2 \quad (2.44)$$

$$C'_4 = C_4, \quad (2.45)$$

where $G_{E\mu\nu} = e^{-\Phi/2} G_{\mu\nu} = e^{-\Phi'/2} G'_{\mu\nu}$. The $SL(2, \mathbb{Z})$ symmetry here makes sure that (p, q) strings, sourced by $\begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$, a doublet of $SL(2, \mathbb{R})$, carry integer charge under the two two-form gauge fields. Recall that in this notation, an F -string has charge $(1, 0)$ and a D -string has charge $(0, 1)$. As mentioned previously, the string coupling g_s is given by the expectation value of the exponential of Φ and the type IIB coupling is (2.36). The above field theory transformation taking the dilation $\Phi \rightarrow -\Phi$ is in fact a symmetry which takes $g_s \rightarrow 1/g_s$ and $\tau \rightarrow -1/\tau$ ($C_0 = 0$): taking a strongly coupled theory to a weakly coupled one. This is called S -duality or strong-weak duality and it relates type IIB superstring theory to itself [12]. It is the first example we encounter of a duality that helps probing the strongly coupled regime of a theory.

2.4 D-branes, Orientifolds and NS5-branes

Branes are extended p spatial dimensional objects in String Theory. Denoted as p -branes, these objects are classified into two categories depending on their tension T_p (energy per unit p -volume) at weak fundamental string coupling g_s [40]:

1. Neveu-Schwarz (NS) or solitonic branes: if the tension behaves like $1/g_s^2$
2. Dirichlet or D-branes: if the tension behaves like $1/g_s$

In the limit $g_s \rightarrow 0$, the above criteria indicate that Dirichlet branes are lighter than NS-branes.

2.4.1 Dirichlet-branes

Dirichlet p -branes [66], denoted Dp -branes are objects stretched along the hyperplane parametrized by (x^1, \dots, x^p) and are point-like in the directions (x^{p+1}, \dots, x^9) . Their defining property is that open strings with Neumann boundary conditions for (x^0, \dots, x^9) can have one of their ends ending on Dp -branes whereas open strings with Dirichlet boundary conditions in (x^{p+1}, \dots, x^9) have both their ends starting and finishing on Dp -branes [40]. We will see below that Dp -branes are sourced by Ramond-Ramond $(p+1)$ -forms potentials in both type IIA and type IIB string theory. As alluded to above, the tension of Dp -branes is given by

$$T_p = \frac{1}{g_s l_s^{p+1}}, \quad (2.46)$$

where l_s is the fundamental string scale. D -branes are BPS objects which preserve half of the thirty two supercharges of type II string theory. In particular, they preserve supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0 \Gamma^1 \dots \Gamma^p \epsilon_R. \quad (2.47)$$

The low energy worldvolume theory on a Dp -brane is a $p+1$ dimensional field theory, the action of which is given by the $p+1$ dimensional Born-Infeld action. Expanding the square root in the latter and keeping only the bosonic part we obtain the following

action [40] on the worldvolume of Dp -branes:

$$S = \frac{1}{g_{SYM}^2} \int d^{p+1}x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{l_s^4} \partial_\mu X^I \partial^\mu X_I \right), \quad (2.48)$$

where the $U(1)$ gauge coupling g_{SYM} on the brane is given by [40]:

$$g_{SYM}^2 = g_s l_s^{p-3}. \quad (2.49)$$

The low energy worldvolume of Dp -branes describes the dynamics of the ground states of open Dirichlet strings. The massless spectrum includes a $p+1$ -dimensional $U(1)$ gauge field $A_\mu(x^\nu)$, $9-p$ scalars $X^I(x^\mu)$ which parametrize the fluctuations transverse to the Dp -branes and some fermions. Here $I = p+1, \dots, 9$ and $\mu = 0, \dots, p$. At high energy, one needs to decouple the massless gauge theory degrees of freedom from gravity and massive string modes if one wants to study SYM on the brane. To do that, we send $l_s \rightarrow 0$ while holding g_{SYM} fixed (this decouples gravity from the action, returning only the open string modes on the brane) [40]. We obtain the following three cases:

- $g_s \rightarrow 0$ for $p < 3$
- $g_s \rightarrow \infty$ for $p > 3$
- g_{SYM} independent of l_s for $p = 3$

The limit $l_s \rightarrow 0$ in the latter case describes a $U(1)$ gauge theory in $\mathcal{N} = 4$ SYM in $3+1$ dimensions. More generally, the theory in the UV behaves as $p+1$ -dimensional SYM for $p \leq 3$. The generalization of (2.48) to N_c parallel Dp -branes is given by the following bosonic $p+1$ kinetic term and potential for the gauge field A_μ and the adjoint scalar X^I :

$$\mathcal{L}_{kin} = \frac{1}{g_{SYM}^2} \text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{l_s^4} \mathcal{D}_\mu X^I \mathcal{D}^\mu X_I \right), \quad (2.50)$$

$$V \sim \frac{1}{l_s^8 g_{SYM}^2} \sum_{I,J} \text{Tr} [X^I, X^J]^2, \quad (2.51)$$

where

$$\mathcal{D}_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I], \quad (2.52)$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu], \quad (2.53)$$

and where the Coulomb branch of the $U(N_c)$ $p+1$ -dimensional SYM theory (4d $\mathcal{N} = 4$ SYM if $p = 3$) therein is parametrized by the flat directions of the above potential¹.

2.4.2 Nonabelian gauge groups from stacks of Dp-branes

As we have just seen, Dp -branes are remarkable objects which can introduce non-abelian gauge theories in string theory. The mechanism responsible of this is the Chan-Paton factor. In the presence of a stack of N_c parallel Dp -branes, the scalars X^I (2.48) turn into $N_c \times N_c$ matrices transforming in the adjoint representation of the gauge group $U(N_c)$. The diagonal component of X^I as well as the N_c massless gauge fields in the Cartan subalgebra of $U(N_c)$ correspond to open strings with both ends ending on the same Dp -brane. The off-diagonal components of X^I and the charged gauge bosons corresponds to strings with endpoints lying on different branes. The (i, j) and (j, i) matrix element of X^I and A_μ are associated with the two orientations of a fundamental string whose endpoints are connected to the i^{th} and j^{th} Dp -branes [40]. In summary, (four-dimensional $\mathcal{N} = 4$) SYM theory with $U(N_c)$ gauge group- 16 supercharges- arise on the low energy worldvolume of a stack of N_c parallel Dp -branes (D3-branes) as a result of the ground states of open strings ending on Dp -branes. We will see in a section below how “webs” of branes can reduce the amount of supersymmetry, focusing on process leading to $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 1$.

¹ Equations (2.50) to (2.53) can be obtained by taking the symmetric trace of the Born-Infeld action.

2.4.3 Orbifolds and Orientifolds

Orbifolds are a class of compactification objects in string theory for which the metric is explicitly known. Roughly speaking, orbifolds are singular spaces defined as quotient spaces of the form X/G where X is a smooth manifold and G a discrete isometry group. A point $g \in X/G$ consists of an orbit of points on the manifold X . Recall that an orbit of $x \in X$ is a point and all of its images under the action of the group G . Singularities on X/G arise when nontrivial group elements leave points in X invariant. The orbifold X/G is locally indistinguishable from the manifold X at nonsingular points [12].

Example of such noncompact space obtained by identifying spacetime coordinates under the reflection of k coordinates $X^k \rightarrow -X^k$ is given by $\mathbb{R}^k/\mathbb{Z}_2$ whereas the compact space version obtained under the same identification on a k -torus leads to T^k/\mathbb{Z}_2 . The former orbifold has k singularities whereas the latter has 2^k fixed points.

While orbifolds preserve the orientation of strings, orientifolds denoted by O_p -planes are generalization of \mathbb{Z}_2 orbifolds fixed plane to non-oriented strings. Namely, they act on spacetime coordinates and reverse the orientation of the string. For example, a \mathbb{Z}_2 O_p -plane extending along (x^1, \dots, x^p) acts as $x^I(z, \bar{z}) \rightarrow -x^I(\bar{z}, z)$ for $I = p+1, \dots, 9$ where z, \bar{z} parametrizes the string worldsheet with $z = e^{\tau+i\sigma}$. As an example, consider the orientifold T^2/\mathbb{Z}_2 in type IIB. The \mathbb{Z}_2 transformation is defined as $(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ where $(-1)^{F_L}$ changes the sign of all the Ramond sector states on the left, Ω denotes the orientation reversal transformation (exchanges the left and right moving modes on the worldsheet) while \mathcal{I}_2 acts on the torus by inverting the sign of both the coordinates of the torus [72].

Orientifolds break the same 16 supercharges of susy as a parallel Dp -brane would. Their sole presence modifies the transverse space by replacing R^{9-p} by R^{9-p}/\mathbb{Z}_2 . This has for consequence to generate mirror \mathbb{Z}_2 images of objects outside the orientifold

plane if one still want to work in R^{9-p} space². In particular, D -branes far from the Op -plane acquire mirror brane images.

As shown in the previous section, Dp -branes are charged under RR $(p+1)$ -form potential in type II. Orientifolds carry charge under the same RR $(p+1)$ -form gauge potential as Dp -branes. Denoting the RR charge of Op -plane by Q_{O_p} , it is equal to the RR charge of 2^{p-4} Dp -branes or 2^{p-5} pairs of Dp -brane and its mirror. The RR charge of D_p -branes is denoted by Q_{D_p} . We summarize here some of the properties of the branes encountered so far in type II string theory[40]:

- Dp -branes: charged under RR $(p+1)$ -form potential in type II
- Type IIA: p even \rightarrow Dp -branes with $p = 0, 2, 4, 6, 8$
- Type IIB: p odd \rightarrow Dp -branes with $p = -1, 1, 3, 5, 7, 9$
- Dp -brane RR charge Q_{D_p} equals its tension (2.46)
- Orientifold: $Q_{O_p} = \pm 2 \cdot 2^{p-5} Q_{D_p}$

The gauge theories living on a stack of N_c Dp -branes parallel to an Op -plane are:

$$G = Sp(N_c/2) \text{ with } N_c \text{ even} \quad Q_{O_p} = +2 \cdot 2^{p-5} Q_{D_p} \quad (2.54)$$

$$G = SO(N_c) \quad Q_{O_p} = -2 \cdot 2^{p-5} Q_{D_p} \quad (2.55)$$

where the rank of the gauge groups are $[N_c/2]$ and they preserve 16 supercharges.

2.4.4 NS5-branes

Solitonic fivebranes are BPS object preserving half the supersymmetry of the theory. Their tension is given by

$$T_{NS5} = \frac{1}{g_s^2 l_s^6}. \quad (2.56)$$

² and implement appropriate (anti)-symmetrization of states

Type IIA NS-fivebranes stretched along (x^1, \dots, x^5) preserve supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_L, \quad (2.57)$$

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_R, \quad (2.58)$$

whereas type IIB fivebranes have

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_L, \quad (2.59)$$

$$\epsilon_L = -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_R. \quad (2.60)$$

Light fields living on the worldvolume of a type IIA NS5-brane form a tensor multiplet of six-dimensional (2,0) susy. The latter is made of a self-dual $B_{\mu\nu}$ field, five scalars and some fermions. Four of the scalars parametrize the fluctuations of the NS-brane in the transverse directions. The fifth scalar lives on a circle of radius l_s . Close to the NS5-brane, g_s is strongly coupled and the circle on which the fifth scalar lives parametrizes the eleventh direction of M-theory. On the other hand, there exists a vector multiplet living on a single type IIB NS5-brane. It contains a six dimensional gauge field, four scalars and some fermions. Here, all four scalars parametrize the transverse directions fluctuations of the fivebrane. On type IIB fivebrane, the vector field's gauge coupling is given by [40]:

$$g_{SYM}^2 = l_s^2. \quad (2.61)$$

Now that we understand what kind of low energy theory lives on a stack of N_c Dp -branes, one could ask the same question about a stack of type IIA NS-branes. The answer is that the low energy theory describing k parallel type IIA NS5-branes correspond to a non-trivial $5 + 1$ dimensional (2,0) field theory. There exists Dirichlet membranes stretched between the NS5-branes. These Dirichlet membranes are tensionless for coinciding NS5-branes. These string-like low energy excitations are charged under the self-dual $B_{\mu\nu}$ fields. The expectation value of the diagonal piece of the five scalars in the tensor multiplet parametrize the Coulomb branch, the origin of

which is a non-trivial superconformal field theory. $(2,0)$ susy in $d = 6$ also appears on type IIB at A-D-E singularities and also appears on coincident M5-branes once we lift type IIA to M-theory. On the other hand, the low energy worldvolume dynamics on a stack of k parallel type IIB NS5-branes is a $5 + 1$ -dimensional $(1,1)$ $U(k)$ SYM described by (2.50)-(2.51) with gauge coupling (2.61) with $p = 5$. It preserves 16 supercharges. However, this is purely informational and we will not discuss about the gauge theory on a stack of NS5-branes in the present thesis.

2.4.5 Webs of branes

We will see below how webs of branes can allow us to study SYM theories with lower supersymmetry than configurations preserving 16 supercharges. Many branes systems exist which preserve 8 supercharges. Some of them are given below [40]:

- $Dp - D(p + 4)$
- $Dp - D(p + 4) + O_p, O_{p+4}$ -planes
- $Dp - D(p + 2)$
- $NS - Dp$

In all the above cases, the main idea is the same: the amount of susy preserved by a system of branes is found by imposing all the susy condition (2.47, 2.57, 2.59) present in the system on the spinors ϵ . We analyze the $Dp - D(p + 4)$ system because it will be useful later when describing the F-theory embedding (using D3-D7-branes) of $\mathcal{N} = 2$ SYM in $d = 4$. Consider a stack of N_c Dp -branes parallel to a stack of N_f $D(p + 4)$ -branes. As seen previously, each stack preserve respectively half the susy of the theory ie. 16 supercharges. It thus makes sense to think that when present together, they preserve one fourth of the susy of the theory namely 8 supercharges. Let's now make this statement more precise by reviewing an example analyzed in detail in [40]. Consider N_c Dp -branes stretched along the hyperplane parametrized by (x^1, \dots, x^p) and N_f $D(p + 4)$ -branes along (x^1, \dots, x^{p+4}) . Combining both susy conditions we find the following constraint on ϵ_L and ϵ_R :

$$\epsilon_L = \Gamma^0 \Gamma^1 \dots \Gamma^p \epsilon_R = \Gamma^0 \Gamma^1 \dots \Gamma^{p+4} \epsilon_R. \quad (2.62)$$

First, we notice that the sole knowledge of ϵ_R fixes ϵ_L . Second, we realize that the LHS of the above equation simplifies to

$$\epsilon_R = \Gamma^{p+1}\Gamma^{p+2}\Gamma^{p+3}\Gamma^{p+4}\epsilon_R. \quad (2.63)$$

The above combination of gamma matrices which we denote by Γ is traceless and squares to the identity matrix. Thus the constraint on ϵ_R translates to $\epsilon_R = \Gamma\epsilon_R$ with $\Gamma = 1$. The tracelessness condition constraints 8 of the 16 eigenvalues of Γ to equal +1 while the remain 8 eigenvalues are equal to -1 . Thus, eight independent components of ϵ_R out of the 16 are preserved by the constraint. In summary, having applied susy conditions of both types of branes onto ϵ , we find that there are eight independent supercharges preserved by this given brane configuration, as expected.

The light degrees of freedom on the N_c Dp -branes comprise N_c $(p+1)$ -dimensional gauge fields A_μ generating a $p+1$ dimensional $U(N_c)$ gauge theory, $9-p$ scalars in the adjoint representation of $U(N_c)$ and some fermions. Recall that Dp -branes have $9-p$ transverse directions whereas $D(p+4)$ branes have $5-p$ transverse directions. Consequently, $5-p$ adjoint scalars of the Dp -brane naturally parametrize the fluctuations of the Dp -branes transverse to the $D(p+4)$ -branes. With the gauge field, these scalars form a vector multiplet while the remaining 4 adjoint scalar parametrize an adjoint hypermultiplet. The light degrees of freedom on $D(p+4)$ branes are the same as those just mentioned provided we replace N_c by N_f and p by $p+4$.

Strings stretched between the two stacks of branes are seen as N_f flavors in the fundamental representation of $U(N_c)$ from the point of view of the Dp -branes. They correspond to N_c pointlike defects in the fundamental representation of $U(N_f)$ from the $D(p+4)$ -branes perspective. From the point of view of Dp -branes, the $U(N_f)$ symmetry is a global symmetry, the only dynamical fields generated from $D(p+4)$ -branes being the N_f flavors. The positions of the $D(p+4)$ -branes in (x^{p+5}, \dots, x^9) correspond to the masses of the N_f fundamentals flavors labelled by \vec{m}_i with $i = 1, \dots, N_f$. These corresponds to couplings in the worldvolume theory of Dp -branes. The locations \vec{x}_a with $a = 1, \dots, N_c$ of the Dp -branes in the transverse space (x^{p+5}, \dots, x^9)

are associated with the expectation values of the adjoint scalars \vec{X} of $U(N_c)$. Their expectation value parametrize the Coulomb branch of the $U(N_c)$ gauge theory with $\vec{x}_a = \langle \vec{X}_{aa} \rangle$. The \vec{x}_a correspond to moduli on the worldvolume theory of Dp -branes. The expectation values of the adjoint hypermultiplet of $U(N_c)$ are parametrized by the position of Dp -branes parallel to $D(p+4)$ -branes in $(x^{p+1}, \dots, x^{p+4})$. As a general rule: the location of the heavy $D(p+4)$ -branes correspond to couplings on the worldvolume of Dp -branes whereas the location of the light Dp -branes are moduli on the same worldvolume [40]. A Dp -branes inside a $D(p+4)$ -branes can be thought of as a small instanton. A Dp -brane embedded in a stack of N_f $D(p+4)$ -branes is a small 4d $U(N_f)$ instanton which can reach finite size. The full Higgs branch of the theory is parametrized by the moduli space of N_c instantons in $U(N_f)$.

Since orientifolds O_p -plane preserve the same supercharges as Dp -branes, adding O_p -planes and/or O_{p+4} -planes to the $Dp - D_{p+4}$ branes system does not further break supersymmetry and thus still preserves 8 supercharges. As seen previously, the presence of O_p -plane generate either $SO(N_c)$ or $Sp(N_c/2)$ gauge theory on the stack of N_c Dp -branes. In the presence of O_p -plane, the global symmetry of a theory preserving 8 supercharges with N_f flavors becomes $Sp(N_f/2)$ or $SO(N_f)$ instead of the original $U(N_f)$ symmetry. We will reach the same conclusion in Chapter 3 while studying F-theory and making use of Tate's algorithm. A construction involving orthogonal O_{p+4} -plane (breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ susy) was analyzed in Gimon-Polchinski [37] and was embedded in F-theory in [27].

Systems of parallel NS5-branes with Dp -branes stretched between them e.g D4-branes in type IIA, D3-branes in type IIB, also preserve 8 supercharges. Their physics completely capture that of $\mathcal{N} = 2$ $d = (3+1)$ $U(N_c)$ SYM field theory [79] and $\mathcal{N} = 2$ $U(N_c)$ SYM $d = 2 + 1$ [50] respectively. Adding flavor branes -say D6-branes or D5-branes respectively, does not break further supersymmetry. In their proper limit of validity, these rich branes constructions capture all the moduli of these field theories [40].

It is possible to further break susy from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by considering system in which some branes span orthogonal directions. The resulting brane construction can

preserve 4 supercharges instead of 8. A famous example is that of two orthogonal NS5-branes with N_c “color” D3-branes stretched between them in the presence of “flavor” D5-branes. This leads to Seiberg duality of $U(N_c)$ $\mathcal{N} = 1$ SQCD in $2 + 1$ dimensions with N_f flavors [31]. The relative angle between the branes govern the couplings between the different complex adjoint scalars and fundamental hypermultiplets into play. For example, rotating one NS5 in Witten’s $\mathcal{N} = 2$ SQCD type IIA model corresponds to giving a mass to the adjoint scalar and explicitly breaking $\mathcal{N} = 2$ to an $\mathcal{N} = 1$ system of orthogonal NS5-branes with D4-branes stretched between them in the presence of D6-branes. These couplings are captured in a superpotential of the form

$$W \equiv \text{tr} \tilde{Q} X \tilde{Q} + \frac{\mu}{2} \text{tr} X^2, \quad (2.64)$$

where Q, \tilde{Q} are the fundamental hypermultiplets, X the adjoint scalar and μ the mass term for the scalar. Susy is enhanced to 8 supercharges when the angle goes to zero, see [40] for details. We will not elaborate on these systems here, saving discussion for relevant models to be analyzed in Chapter 4.

2.5 M-theory

Type IIA and type IIB defined above are valid at small g_s , where perturbation theory applies. We have already seen what happens to type IIB in the large g_s regime namely, S-duality maps type IIB to type IIB so we now focus on type IIA. When g_s becomes large and is outside the regime of perturbative string theory, type IIA grows an eleventh dimension (parametrized by x^{10}) in the form of a circle of size $g_s l_s$. The new 11-dimensional quantum theory which emerges is called M-theory. At low energy, and in flat $1 + 10$ dimensional Minkowski vacuum, the latter is approximated by 11-dimensional supergravity: a classical field theory. The only parameter of M-theory is the eleven dimensional Planck scale l_p : physics is strongly coupled at scales smaller than l_p and well approximated by weakly coupled semiclassical supergravity for scales much larger than l_p [40]. The spectrum of M-theory includes a 3-form potential A_{MNP} with $M, N, P = 0, 1, \dots, 10$ which sources electrically an M-theory membrane called M2-branes while magnetically coupling to an M-theory fivebrane denoted M5-brane.

The tension of Mp -branes is given by the $p + 1$ power of the Planck's length [40]:

$$T_p = 1/l_p^{p+1}. \quad (2.65)$$

These Mp -branes preserve half of the thirty two supercharges of the theory. In particular, an Mp -brane stretched in the hyperplane (x^1, \dots, x^p) with $p = 2, 5$ preserve the supercharges ϵQ with

$$\Gamma^0 \Gamma^1 \dots \Gamma^p \epsilon = \epsilon. \quad (2.66)$$

As mentioned previously, type IIA with a finite string coupling g_s can be thought of as M-theory compactified on $R^{1,9} \times S^1$ (where the radius of S^1 is denoted by R^{10}). In the limit $g_s \rightarrow 0$ and $S^1 \rightarrow 0$ we recover ten dimensional type IIA. The relations below between the M-theory compactification radius R_{10} , l_p , and the type IIA parameters g_s , l_s clarify that the strong coupling limit of type IIA $g_s \rightarrow \infty$ ($R_{10}/l_p \rightarrow \infty$) is given by the $1 + 10$ dimensional Minkowski vacuum of M-theory.

$$\frac{R_{10}}{l_p^3} = \frac{1}{l_s^2}, \quad (2.67)$$

$$R_{10} = g_s l_s. \quad (2.68)$$

The type IIA branes that were described the previous sections all have interpretations in M-theory. We review in Table 2.1 some examples of correspondences that will be useful for our discussion in Chapter 3 and 4. From the table below, one notices that type IIA D4-branes and NS5-branes both emerges from the same object in M-theory, namely M5-branes. D4-branes in $R^{1,9}$ corresponds to M5-brane which wraps the S^1 direction in $R^{1,9} \times S^1$ whereas the NS5-brane on $R^{1,9}$ becomes a transverse M5 brane on $R^{1,9} \times S^1$ whose world volume is pointlike on S^1 . Thus, configurations of parallel NS5-branes with N_c D4-branes stretched between them and generating $\mathcal{N} = 2$ $U(N_c)$ SYM in four dimensions are described by a single fivebrane in M-theory. The worldvolume of the M5-brane is $R^{1,3} \times \Sigma$ with Σ embedded in the four-manifold $Q \cong R^3 \times S^1$. Q is parametrized by the complex coordinates $v = x^4 + ix^5$ and $s = x^6 + ix^{10}$. Imposing $\mathcal{N} = 2$ susy means giving Q a complex structure in which v and s are holomorphic. As a consequence, Σ is a complex smooth Riemann surface with genus g equal to the

Type IIA	M-theory	Charged under
Fundamental string x^1	M2-brane $x^{1,10}$	$B_{\mu 1} = A_{10\mu 1}$
D4-brane x^{01236}	M5-brane $x^{0,1,2,3,6,10}$	$\tilde{A}_{10\mu_1\mu_2\cdots\mu_5}$ ($d\tilde{A} = *dA$)
NS5-brane x^{012345}	M5-brane x^{012345}	$\tilde{A}_{\mu_1\cdots\mu_6}$
D6-branes $x^{0123456}$	KK monopole $x^{0123456}$	$A_\mu = G_{\mu 10}$

Table 2.1: IIA M-theory correspondence. Directions along which the branes are stretched is indicated.

rank of the SYM gauge group. The latter surface is described by a hyperbolic curve which can take the form of a Seiberg-Witten curve [79] or a spectral curves (n-sheeted cover of the genus-g Riemann surface) - usually used in integrable systems [28].

M-theory is of course much richer than what we have explored so far. For, it is believed that all ten dimensional string theories can arise as asymptotic expansions around different vacua of M-theory; dualities connecting the different 10-dimensional theories together. We will not elaborate on this here. We will however present in the next section our first example of a gauge theory that we will embed in string theory in Chapter 3. We thus turn to the presentation of the $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions called Seiberg-Witten theory.

2.6 Seiberg-Witten theory

We start by discussing some generic features of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory preserving 8 supercharges. Later, we will focus on a specific theory named Seiberg-Witten theory with gauge group $SU(2)$ and 4 flavors in the fundamental representation of the gauge group.

Four-dimensional $\mathcal{N} = 2$ supersymmetry means that there are 2 generators of supersymmetry Q_α^A with spinor index α and $A = 1, 2$ with algebra:

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = -2\delta^{AB}P_\mu\Gamma_{\alpha\beta}^\mu, \quad (2.69)$$

$$\{P^\mu, Q_\alpha^A\} = 0. \quad (2.70)$$

Q^1, Q^2 are two Majorana spinors each containing 4 linearly independent supercharges satisfying a reality condition. One can also write them as 2 Weyl spinors with two complex components each. We will use the Majorana notation. P^μ is the spacetime momentum and $\bar{Q} \equiv Q^\dagger \Gamma^0$ from Majorana properties [65]. $\mathcal{N} = 2$ susy theories in $d = 4$ thus have 8 supercharges transforming as 2 copies of $\mathbf{2} + \bar{\mathbf{2}}$ of $Spin(1, 3)$ which is roughly speaking isomorphic to $SL(2, \mathbb{C})$. Every $\mathcal{N} = 2$ susy theories have an $SU(2)_R$ global symmetry acting on the 2 supercharges. In addition, conformal theories have an extra $U(1)_R$ global symmetry under which chiral supercharges have charge ± 1 . $\mathcal{N} = 2$ theories have 3 massless multiplets: the vector multiplet, the hypermultiplet, and the supergravity multiplet. We will be interested in the first two.

The vector multiplet contains the following fields:

$$\begin{array}{ccc} & A_\mu & \\ \lambda_\alpha & & \psi_\alpha \\ & \phi & \end{array} \quad (2.71)$$

Where A_μ is a gauge field, $\lambda_\alpha, \psi_\alpha$ are Weyl fermions and ϕ is a complex scalar. The diamond shape indicates how each line in (2.71) transforms under the global $SU(2)_R$ symmetry: the gauge field and the complex scalar are both singlets under $SU(2)_R$ while the fermions λ, ψ form a doublet transforming in the $\mathbf{2}$ of $SU(2)_R$. All the fields forming the vector multiplet transform in the adjoint representation of the gauge group. Using $\mathcal{N} = 1$ supersymmetry language, the $\mathcal{N} = 2$ vector multiplet decomposes into a $\mathcal{N} = 1$ vector multiplet and a $\mathcal{N} = 1$ chiral multiplet. The $\mathcal{N} = 1$ vector superfield (also called $\mathcal{N} = 1$ vector multiplet) is given, in the notation of [77],

by:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu - i\bar{\theta}^2(\theta\lambda) + i\theta^2(\bar{\theta}\bar{\lambda}) + \frac{1}{2}\theta^2\bar{\theta}^2D, \quad (2.72)$$

and the associate gauge covariant field strength is given by:

$$W_\alpha = \bar{\mathcal{D}}^2(e^{2V}\mathcal{D}_\alpha e^{-2V}), \quad (2.73)$$

where $V = V_a T^a$, $a = 1, \dots, \dim G$. In the equation above, V_a is the $N = 1$ vector multiplet (2.72) with a the index labelling the adjoint representation of the gauge group and T^a are the generators of the gauge group G in the representation R . Also, \mathcal{D} was defined in (2.52) where the scalars are now ϕ . The $\mathcal{N} = 1$ chiral superfield is given by:

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F. \quad (2.74)$$

The low energy Lagrangian describing the $\mathcal{N} = 2$ vector multiplet can be written in terms of $\mathcal{N} = 1$ superspace as [40]:

$$\mathcal{L}_{vec} = \text{Im Tr} \left[\tau \left(\int d^4\theta \Phi^\dagger e^{-2V} \Phi + \int d^2\theta W_\alpha W^\alpha \right) \right], \quad (2.75)$$

where the complex gauge coupling τ is defined as

$$\tau = \frac{\theta}{2\pi} + \frac{i}{g_{SYM}^2}, \quad (2.76)$$

and the trace in (2.75) is over the gauge group. The bosonic part of (2.75) decomposes as a kinetic term given by (2.50) and a potential of the form

$$V \sim \text{Tr}[\phi^\dagger, \phi]^2. \quad (2.77)$$

On the other hand, the $\mathcal{N} = 2$ hypermultiplet are made of

$$\begin{array}{cc} & \psi_q \\ q & \tilde{q}^\dagger \\ & \psi_{\tilde{q}}^\dagger \end{array} \quad (2.78)$$

two Weyl fermions ψ_q and ψ_q^\dagger as well as two $2\dim R$ complex scalar q, \bar{q}^\dagger . In $\mathcal{N} = 1$ superspace language, the $\mathcal{N} = 2$ hypermultiplet decomposes into two $\mathcal{N} = 1$ chiral superfields (multiplets) denoted Q, \tilde{Q} transforming in the representation R, \bar{R} respectively of the gauge group G . Again, the diamond shape of (2.78) reminds us of how the fields transform under $SU(2)_R$. The fermions are singlets under $SU(2)_R$ and carry $U(1)_R$ charge 1 while the scalar components of Q, \tilde{Q} transform as a doublet under $SU(2)_R$ and carry no charge under $U(1)_R$. In $\mathcal{N} = 1$ superspace language, the low energy Lagrangian describing the hypermultiplet is given by [40]:

$$\mathcal{L}_{hyper} = \int d^4\theta \left(Q^\dagger e^{-2V} Q + \tilde{Q}^\dagger e^{-2V} \tilde{Q} \right) + \int d^2\theta \tilde{Q} \Phi Q + c.c. \quad (2.79)$$

When formulating the complete theory

$$\mathcal{L} = \mathcal{L}_{vec} + \mathcal{L}_{hyper}, \quad (2.80)$$

it has a Coulomb branch parametrized by the matrices ϕ satisfying $V = 0$, e.g. $[\phi, \phi^\dagger] = 0$. ϕ in the Cartan subalgebra of the gauge group $\phi = \sum_{i=1}^r \phi_i T^i$ generates $r = \text{rank } G$ complex moduli parametrizing the Coulomb branch leading to a $U(r)$ gauge group. When given a VEV, the gauge group Higgs to $U(1)^r$. In the presence of matter in the fundamental representation of the gauge group, the complex scalars in the hypermultiplet parametrize the Higgs branch. $\mathcal{N} = 2$ susy ensures that the moduli space of vacua is not lifted by quantum effects. However, the metric is modified.

We now turn to a particular $\mathcal{N} = 2$ susy gauge theory with group $SU(2)$. We will not put fundamental matter just yet for simplicity. It was shown by Seiberg and Witten [68] that $\mathcal{N} = 2$ SYM theories in four dimensions with at most two derivatives and four fermions can be solved exactly. By exactly we mean here that while capturing all non-perturbative effects, they still found exact formulas for the metric on the moduli space of vacua as well as for electrons and dyons masses. This surprising property of Seiberg-Witten theory comes from the fact that the theory's dynamics is governed by holomorphic quantities. In fact, Wilsonian effective action

with higher derivative terms are not governed by such holomorphic quantities and would not lead to exact solutions.

The holomorphicity of the quantity \mathcal{F} , a function of the moduli space called the prepotential, allowed Seiberg and Witten to express the $U(1)$ gauge theory completely in terms of the following action [68]:

$$\mathcal{L}_{vec} = \text{Im} \text{Tr} \left[\int d^4\theta \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi_i} \bar{\Phi}_i + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_i \partial \Phi_j} W_\alpha^i W_j^\alpha \right], \quad (2.81)$$

where Φ is the $\mathcal{N} = 1$ chiral multiplet with scalar component ϕ inside the $\mathcal{N} = 2$ vector multiplet while W^α is the $\mathcal{N} = 1$ vector multiplet inside the $\mathcal{N} = 2$ vector multiplet in $\mathcal{N} = 1$ superspace language. Since the quantum corrections on the moduli space prevent the $SU(2) = Sp(2)$ classical gauge group from enhancing, the gauge group is $U(1)$ everywhere on the quantum moduli space and we don't need to go beyond the $U(1)$ action shown above. Gauge symmetry group enhancement point on the classical moduli space correspond to two points where the monopole and dyon are massless. The physical meaning of these singularities can be understood as follows: imagine you are at an energy scale Λ , some massive states exist at that energy scale so you integrate them out. You then flow to the low energy effective action. If you see singularities there, it means that you've integrated out massive states at energy Λ which became massless at low energy. Singularities in the low energy effective Wilsonian action thus mean that massless states were integrated out. Underlying monodromies on the moduli space tell us what states have been integrated out. This information is captured by an elliptic curve non-trivially fibered over the moduli space. More on this in a bit. Coming back to the prepotential function \mathcal{F} , the latter defines the low energy $U(1)^r$ gauge coupling matrix τ_{ij} which itself parametrizes the metric on the moduli space

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial \phi_i \partial \phi_j}, \quad (2.82)$$

$$ds^2 = \text{Im} \tau_{ij} d\phi_i d\bar{\phi}_j = \text{Im} \mathcal{F}''(\phi) d\phi d\bar{\phi}. \quad (2.83)$$

Demanding $\text{Im } \tau(\phi) > 0$ guaranties the existence of a well-defined positive definite metric everywhere on the moduli space. This was one of the crucial ingredient leading to exact solution in Seiberg-Witten theory. The other essential piece of physics to Seiberg-Witten's result is that the holomorphic prepotential \mathcal{F} is itself constrained by the weakly coupled limit of the $\mathcal{N} = 2$ theory. This can be understood as follows: if one compares the $U(1)$ Wilsonian effective action of $\mathcal{N} = 2$ SYM (2.81) to the $\mathcal{N} = 2$ vector multiplet lagrangian (2.75), one sees that classically, the prepotential is given by the following quadratic function [40, 68]:

$$\mathcal{F}_0 = \frac{1}{2}\tau_0\Phi_i\Phi^i = \frac{1}{2}\tau_0\mathcal{A}^2, \quad (2.84)$$

where τ_0 is the bare coupling constant. After adding the 1-loop correction - in the absence of fundamental matter- Seiberg showed [70] that the form of the prepotential is [40, 68]:

$$\mathcal{F}_1 = \frac{i}{4\pi} \sum_{\vec{\alpha} > 0} (\vec{\alpha} \cdot \vec{\Phi})^2 \log \frac{(\vec{\alpha} \cdot \vec{\Phi})^2}{\Lambda^2} = \frac{i}{2\pi} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2}, \quad (2.85)$$

where $\vec{\alpha}$ are the positive root of the Lie algebra of the gauge group G , \mathcal{A} is the $\mathcal{N} = 2$ vector multiplet and Λ is the dynamically generated scale. The logarithm breaks $U(1)_R$ symmetry and is related to the 1-loop beta function. A non-renormalization theorem assures that higher order perturbative corrections are absent. However, there exists an infinite series of non-perturbative corrections coming from instantons corrections. This series falls off algebraically at large Φ but is important at small Φ . The full prepotential function is thus given by

$$\mathcal{F} = \frac{i}{2\pi} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathcal{A}} \right)^{4k} \mathcal{A}^2, \quad (2.86)$$

where the last term is the instanton contribution and where k instantons contribute to the k' th term. Note that for cases where the beta-function vanishes such as in $\mathcal{N} = 4$ SYM theories, the classical prepotential is exact e.g has no perturbative or non-perturbative corrections. The last building block on which Seiberg-Witten

theory lies on is the generalization of Montonen-Olive $SL(2, \mathbb{Z})$ duality [62] in $\mathcal{N} = 4$ superconformal field theories to $\mathcal{N} = 2$ supersymmetric ones. Seiberg and Witten showed in [68] that there also exists an $SL(2, \mathbb{Z})$ symmetry governing $\mathcal{N} = 2$ SYM theories which interchanges strongly coupled gauge theory for weakly coupled one, provided on interchanges the electrically stable states for magnetically charged ones. This duality exchanges the $\mathcal{N} = 2$ $U(1)$ action for its dual Lagrangian given by

$$\mathcal{L}_{vec}^D = \text{Im} \text{Tr} \left[\int d^4\theta \mathcal{F}'_D(\Phi_D) \bar{\Phi}_{Di} + \frac{1}{2} \int d^2\theta \mathcal{F}''_D(\Phi_D) W_{\alpha D}^i W_{jD}^\alpha \right], \quad (2.87)$$

where the subscript D denotes electric-magnetic dual variable and where we used the following relation:

$$\phi_i^D = \frac{\partial \mathcal{F}}{\partial \phi^i}. \quad (2.88)$$

Accordingly, the metric on moduli space can thus be written in the following compact form:

$$ds^2 = \text{Im} (d\phi^D d\bar{\phi}). \quad (2.89)$$

Lastly, the prepotential determines the mass of BPS states of the theory. For BPS saturated states with electric charges e_i and magnetic charges m^i with $i = 1, \dots, r$ under the r unbroken $U(1)$ gauge fields, the supersymmetry algebra yields the following mass

$$M = \sqrt{2}|Z|, \quad (2.90)$$

with the central charge Z given by

$$Z = \phi^i e_i + \phi_i^D m^i, \quad (2.91)$$

which can be written in term of the prepotential by using (2.88). Therefore, by determining exactly the prepotential \mathcal{F} for the four-dimensional $\mathcal{N} = 2$ SYM with gauge group $SU(2)$ both with and without matter, Seiberg and Witten completely solved the theory. In addition, they found that τ_{ij} is the period matrix of a Riemann surface with genus one; showing in the process that the moduli space of vacua of this theory is parametrized by the complex structure of an auxiliary two dimensional Riemann

surface. The elliptic curve mentioned previously which captured the singularities on the moduli space parametrize the aforementioned Riemann surface. We will see the appearance of such a Riemann surface again when discussing about F-theory. Before we show how F-theory provides a geometric understanding of all the features of Seiberg-Witten $\mathcal{N} = 2$ SYM theory in 4 dimensions, we turn to a more complicated field theory with the higher rank gauge group $SU(3)$ exhibiting a duality called Argyres-Seiberg duality. We will later see how to embed both Seiberg-Witten theory and its higher rank generalizations in F-theory.

2.7 Argyres-Seiberg duality

We would like to now discuss about a more complicated superconformal gauge theory: $\mathcal{N} = 2$ SYM theory with gauge group $SU(3)$ with 6 fundamental flavors. This theory leads to a duality called Argyres-Seiberg duality [7]. This is an extension of S-duality (strong-weak duality) of $\mathcal{N} = 4$ supersymmetric gauge theories (also called Olive-Montonen duality [62]) to the larger class of $\mathcal{N} = 2$ superconformal gauge theories and will be the corner stone of Chapter 3.

S-duality in $\mathcal{N} = 4$ superconformal field theories answers the following question: what happens when the gauge coupling constant g inside the complex coupling $\tau = \theta/(2\pi) + (4\pi i)/g^2$ becomes infinite? The answer for four-dimensional $\mathcal{N} = 4$ SYM theories is that the theory turns into a weakly coupled gauge theory, not necessarily with the same gauge group though. For simply-laced gauge groups, where theory is self dual, this duality is expressed as an equivalence between the theory at different couplings $\tau \cong -1/\tau$. The periodicity of $\theta \rightarrow \theta + 2\pi$ leads to further identification, namely $\tau \cong \tau + 1$ [7].

The two symmetries of the complex coupling τ generate an $SL(2, \mathbb{Z})$ group of identifications whose fundamental domain in the space of couplings is bounded away from infinite coupling given by $\text{Im}\tau = 0$ see figure (2.1).

The case of infinite coupling in four dimensional scale-invariant $\mathcal{N} = 2$ SYM theories is more subtle. For $\mathcal{N} = 2$ $SU(2)$ SQCD with four massless hypermultiplets

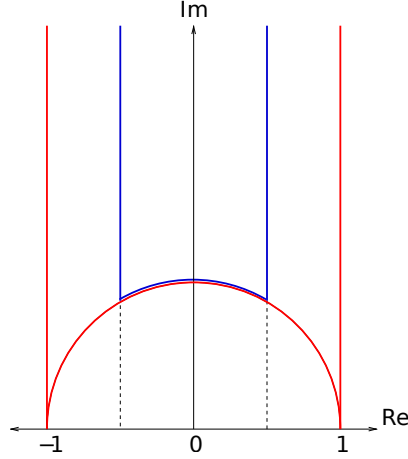


Figure 2.1: Fundamental domain in τ for $SL(2, \mathbb{Z})$ (blue) and in $\tilde{\tau} \equiv 2\tau$ for $\Gamma^0(2)$ (red). Edges of the domains are identified under reflection by $\text{Re}\tau = 0$. Figure extracted from [7] and modified.

in the fundamental representation of the gauge group, it was shown in [68] that Olive-Montonen duality goes through and that there is an $SL(2, \mathbb{Z})$ S-duality. However, this is not the case for higher rank gauge group. In particular, for the $\mathcal{N} = 2$ SYM theory with $SU(3)$ gauge group with 6 massless fundamental hypermultiplets it was shown in [5] that the S-duality group is $\Gamma^0(2) \subset SL(2, \mathbb{Z})$, generated by $\tilde{\tau} \cong \tilde{\tau} + 2$ and $\tilde{\tau} \cong -1/\tilde{\tau}$ where $\tilde{\tau} \equiv 2\tau$. As seen in figure (2.1), $\text{Im}\tilde{\tau}$ can equal to zero, meaning that the theory contains points of infinite coupling in its moduli space.

Argyres and Seiberg studied the physics at the infinite coupling points of the superconformal $\mathcal{N} = 2$ theory with gauge group $SU(3)$ and provided an M-theory description of the phenomenon. Since we will want to embed this duality as well as its extension - called Gaiotto duality - in F-theory, we will not present here the M-theory description they found. What we will present however is the field theory answer that Argyres and Seiberg found to the following question: what happens when we take the marginal gauge coupling g to infinity in the four-dimensional $\mathcal{N} = 2$ SYM theory with gauge group $SU(3)$ in the presence of 6 massless fundamental hypermultiplets.

The answer is [7]:

$$SU(3)_{\text{w}/6} \cdot (\mathbf{3} \oplus \bar{\mathbf{3}}) = SU(2)_{\text{w}/(2 \cdot \mathbf{2} \oplus \text{SCFT}_{\mathbf{E}_6})}. \quad (2.92)$$

It reads as follows: the gauge theory with gauge group $SU(3)$ coupled to 6 massless hypermultiplets in the fundamental representation of the gauge group with a coupling f is equivalent to an $SU(2)$ gauge theory (with coupling \tilde{f}) with one massless fundamental hypermultiplet. The $SU(2)$ is also coupled to an isolated rank 1 SCFT with flavor symmetry E_6 [7]³. The way the $SU(2)$ gauge group is coupled to the SCFT is by gauging the $SU(2)$ inside the maximal subgroup $SU(2) \times SU(6) \subset E_6$. Thus, effectively, the flavor symmetry seen by the $SU(2)$ gauge group is given by $U(6) = U(1) \times SU(6)$ where the $U(1)$ comes from the massless hypermultiplet and the $SU(6)$ from what's left of the E_6 flavor symmetry after the $SU(2)$ is gauged. Only in the limit of zero coupling (when $SU(2)$ decouples from the rank 1 SCFT) is the E_6 the full flavor symmetry of the theory. Argyres-Seiberg duality maps an infinite coupling in f to zero coupling in \tilde{f} where $f \sim e^{i\pi\tilde{\tau}} \rightarrow 0$ at weak coupling and $f \rightarrow 1$ at infinite coupling. A quick check of the duality (2.92) is through the matching of the ranks and flavor groups on both sides [7]. The rank - real dimension of the Coulomb branch - of the $SU(3)$ gauge group of the left hand side of (2.92) is equal to 2. On the other hand, both the $SU(2)$ gauge group and the E_6 SCFT on the right hand side of (2.92) have rank 1 matching that of $SU(3)$. The 6 massless hypermultiplets on the left hand side of (2.92) contribute to an $U(6)$ flavor group. We have already establish that the E_6 SCFT contributes an $SU(6)$ flavor symmetry from the way the $SU(2)$ is gauged inside the maximal subgroup of E_6 . Adding the $U(1)$ flavor from the hyper on the right hand side, this matches the flavor symmetry of the left hand side, thus successfully checking the duality.

Since the topics of Chapter 3 include the embedding of Argyres-Seiberg duality and its generalization in F-theory, we begin the next section by introducing F-theory.

³ What is meant here by “isolated SCFT” is an SCFT with no marginal coupling on its own. The rank of an isolated $\mathcal{N} = 2$ SCFT is equal to the complex dimension of its Coulomb branch.

2.8 F-theory

As we have seen a couple of times now, dualities both in field theories and string theory have allowed us to probe the strong coupling regime of many corners of string theory. M-theory with its strong-weak duality to type IIA has played a significant role in this endeavour. However, there are theories, such as type IIB which have a less natural interpretation in M-theory. Surely, one can understand the $SL(2, \mathbb{Z})$ invariance of 10d type IIB by first compactifying the theory to 9d and then compare it with a T^2 compactification of 11d M-theory. One then finds that the $SL(2, \mathbb{Z})$ is interpreted as a symmetry of the torus but one recovers 10d type IIB only in the limit where the T^2 has zero area- see [40] for more details. Wanting to associate a geometric meaning of the $SL(2, \mathbb{Z})$ invariance of type IIB and desiring a strongly coupled dual theory to type IIB (in the same way M-theory is the strongly coupled dual to type IIA), Vafa proposed in 1996 the F-theory [76].

F-theory is a 12-dimensional theory which, once compactified on a 4-dimensional K3 manifolds, gives rise to an 8 dimensional susy field theory preserving half of the supersymmetry. Vafa argued that the uncompactified 8-dimensions correspond to the worldvolume of 7-branes.

By definition, an elliptically fibered manifold (a manifold that admits elliptic fibration) is a manifold M that has the structure of a fiber bundle whose fiber is a two dimensional torus at every points of a base which is some manifold B . F-theory compactified on a manifold M corresponds to type IIB compactified on the manifold B . More precisely, F-theory compactified on an elliptically fibered K3 manifold corresponds to type IIB compactified on S^2 (where the S^2 was obtained after compactifying CP^1 by putting on it 24 $D7$ -branes) [72]. Under 10 dimensional $SL(2, \mathbb{Z})$ strong-weak duality, the type IIB gauge coupling $\tau = C_0 + ie^{-\phi}$ transforms in the same way as the modulus of the torus. In fact, the complex structure moduli of the elliptic fiber in F-theory is dynamical and under this map from F-theory to type IIB, it corresponds to the gauge coupling (axion-dilaton modulus) of type IIB which captures the aforementioned dynamic by depending holomorphically on the complex coordinates parametrizing CP^1 . If z and \bar{z} are the coordinates of S^2 then

τ only depends on z or \bar{z} . (Here, holomorphicity comes from looking at vacuum solution of type IIB and demanding that the solution of the low energy Lagrangian preserve 1/2 susy). Since the antisymmetric NS-NS and RR tensors are interchanged under $SL(2, Z)$ transformation (and thus not invariant), there are set to zero in this discussion (i.e set to zero when solving for vacuum solution of type IIB) [76]. To understand how τ depends on z , one can start by writing how the torus depends on z . The equation for the torus as a function of z is given by the following elliptic curve, also called Weierstrass equation [72, 76]:

$$y^2 = x^3 + f^8(z)x + g^{12}(z), \quad (2.93)$$

where f^n, g^m are degree n, m polynomial in z while $x, y, z \in \mathbb{CP}^1$. The above equation defines an elliptically fibered K3 surface where there is a torus at each point on \mathbb{CP}^1 parametrized by the coordinate z . $\tau(z)$, the modular parameter of the torus is given by the ratio:

$$j(\tau(z)) = \frac{4 \cdot (24f)^3}{27g^2 + 4f^3}, \quad (2.94)$$

with

$$j(\tau) = \frac{(\theta_1^8(\tau) + \theta_2^8(\tau) + \theta_3^8(\tau))^3}{\eta(\tau)^{24}}, \quad (2.95)$$

where the theta functions satisfying Jacobi's identity were defined in [69] to be

$$\theta_1(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \quad (2.96)$$

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} \quad (2.97)$$

$$\theta_3(\tau) = q^{\frac{1}{2}n^2}, \quad (2.98)$$

with $q = e^{2\pi i \tau}$ and $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - q^n). \quad (2.99)$$

Positions where the torus degenerates correspond to points where the discriminant of the above equation vanishes

$$\Delta \equiv 4f^3 + 27g^2. \quad (2.100)$$

From the type IIB perspective, $\tau = C_0(z) + ie^{-\phi(z)}$ and the 7-branes of type IIB transverse to CP^1 are located at the zeroes of the above discriminant. Since generically there are 24 zeroes of Δ , there are 24 of these 7-branes on CP^1 . Let z_i be a zero of Δ , then for z near z_i , (2.94) and (2.100) lead to [72]:

$$j(\tau(z)) \sim \frac{1}{z - z_i}. \quad (2.101)$$

Thus, up to $SL(2, \mathbb{Z})$ transformation, the torus modular parameter $\tau(z)$ reads [72]:

$$\tau(z) \sim \frac{1}{2\pi i} \ln(z - z_i). \quad (2.102)$$

We now observe the following: there are 24 (p, q) 7-branes each carrying some RR charges on a compact S^2 manifold. The flux of the branes charge has nowhere to go. A legitimate question to ask is if this picture is inconsistent with Gauss' law? The answer to this puzzle, provided by Sen in [72], is as follows: in the weak string coupling, 16 of the 7-branes are $D7$ -branes with charge $(1, 0)$. They are in the presence of 4 O_7 -planes carrying -4 charges of D_7 branes (2.55) and thus cancelling the total RR charges on S^2 . At strong coupling, these O_7 -plane split into 2 (p, q) 7-branes each: 1 with charge $(0, 1)$ and the other with charge $(1, -1)$. Thus, in total, the 24 original (p, q) 7-branes are split into 16 $D7$ -branes, 4 $(0, 1)$ 7-branes and 4 $(1, -1)$ 7-branes⁴. As we will see next, Sen's understanding of F-theory is going to play a crucial role in embedding four-dimensional Seiberg-Witten's $\mathcal{N} = 2$ SYM theory in F-theory.

⁴ (p, q) seven-branes are related to each other by $SL(2, \mathbb{Z})$ transformations. This explains why the dyon, in the literature, is sometimes written with charge $(2, 1)$ instead of $(1, -1)$.

2.9 F-theory embedding of $\mathcal{N} = 2$ SUSY $d = 4$

It was shown by Sen [72] and later rendered even more precise by Banks-Douglas-Seiberg [11] that a single D3-brane in the background of an orientifold 7-plane (with 4 $D7$ branes) reproduces Seiberg-Witten theory (with fundamental matter) [68, 69].

In Seiberg-Witten field theory without matter content⁵, the Coulomb branch- loosely speaking referred to as the moduli space of vacua- is a 2 real dimensional plane parametrized by a gauge invariant modulus called u . The u -plane can also be written in terms of adjoint complex scalar and thus its complex dimension corresponds to the rank of the gauge group. Since $G = SU(2) \cong Sp(2)$ has rank 1, recall $SU(r+1)$ has rank r , the $\dim_{\mathbb{C}}(\text{Coulomb branch}) = 1$. We will embed this field theory in string theory by considering type IIB on $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$. The moduli space of vacua, parametrized by $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{R}^4$ contains many branches. Amongst others, there is the one complex dimensional Coulomb branch parametrized by the $v \equiv x^4 + ix^5$ direction with geometry $\mathbb{R}^2/\mathbb{Z}_2$ and a Higgs branch along x^{6789} with geometry \mathbb{R}^4 . The \mathbb{R}^{0123} spans Minkowski space. When adding sufficient number of 7-branes- in total 24- the moduli space $\mathbb{R}^2/\mathbb{Z}_2$ compactifies to $\mathbb{T}^2/\mathbb{Z}_2 \cong \mathbb{CP}^1$. This leads to F-theory on an elliptically fibered $K_3 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$. Recall that in the latter compact case, the RR fluxes of 7-branes have not enough non-compact transverse directions to escape. To cancel the RR charge, we consider, classically, 1 O_7 -plane with 4 parallel $D7$ -branes and their mirror. The type IIB geometry we will be concerned with at the moment is $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$ with branes spanned along the following spacetime directions:

$$D3 : \quad 0123 \tag{2.103}$$

$$O_7 : \quad 01236789 \tag{2.104}$$

$$D7 : \quad 01236789. \tag{2.105}$$

⁵ In Seiberg Witten theory with matter, the moduli space of vacua is six real dimensional and parametrised by one complex scalar in the $N = 2$ vector multiplet and two complex scalars in the $N = 2$ hypermultiplet. In Seiberg Witten theory without matter, there is no hypermultiplet so the moduli space of vacua is the Coulomb branch.

The RR charges, given in (2.55) $Q_{O_7} = -8Q_{D_7}$ thus cancel since the O_7 plane carry -4 charges of $D7$ -branes (-8 charge of $D7$ -brane and their mirror). The full classical configuration generating Seiberg-Witten theory is that of $1 D3 - 1 D3_m, 1 O_7 - 4 D7 - 4 D7_m$. Recall that in the presence of the O_7 -plane, the D3-brane also gets a mirror D3-brane denoted here by $D3_m$. Modulo the $D7$ -branes, this classical setup is depicted in figure (2.2). The neutral gauge boson corresponds to the ground state of

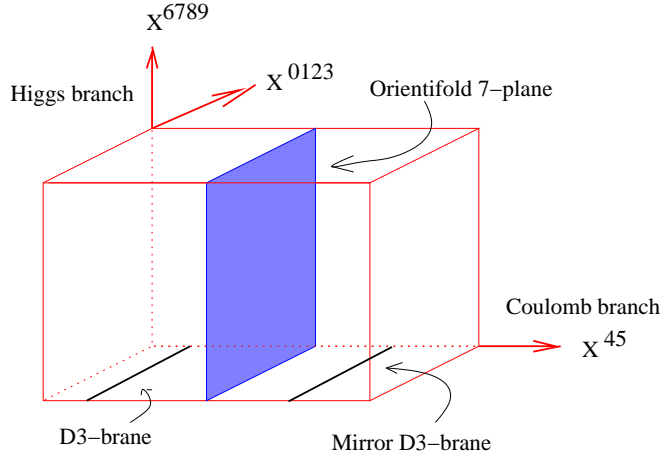


Figure 2.2: In F theory a single D3-brane probing an orientifold seven-plane background maps to the classical picture of $Sp(2)$ Seiberg-Witten theory.

an open string with both ends on the D3-brane. On the other hand, charged gauge bosons correspond to the ground states of open strings stretched between the D3-brane and its mirror. The effective mass of the matter i^{th} quark ($i = 1, \dots, 4$) is given by the relative distance between the i^{th} $D7$ -brane and the D3-brane. Similarly for the mirror $D7$ -branes and the mirror D3-brane. We can make this discussion more precise since all the objects in (2.103)-(2.105) are pointlike along the Coulomb branch. Let the O_7 -plane be at the origin of the Coulomb branch $v = 0$, put the four $D7$ -branes and their mirrors at $m_i, -m_i$ and let the $D3$ -brane and its image be at $v, -v$. The $D7$ -branes give rise to $N_f = 4$ fundamental hypermultiplets Q_i, \tilde{Q}_i . The $D7$ -branes positions m_i correspond to the bare masses of quarks. The effective mass of the quarks are given by $m_i - v$ and $m_i + v$ [40]. The mass of the charged W^\pm is $2|v|$ in string units. When the $D3$ -brane and its mirror are coinciding with the

orientifold plane, i.e. when $v = 0$, the charged gauge boson becomes massless and the gauge group is classically enhanced to $Sp(2) \cong SU(2)$. When the $D3$ -brane and its mirror are away from $v = 0$, the gauge boson picks up a mass which higgses the gauge group to $U(1)$. When $D7$ -branes and their mirrors coincide with the O_7 -plane, quarks become massless and the gauge symmetry on the 7-branes is enhanced to $SO(8)$. When the $D7$ -branes are away from the O_7 -plane, the symmetry group is broken to $U(1)^4$. From the point of view of $1 + 3$ dimensional physics on the $D3$ -branes, the aforementioned $SO(8)$ symmetry is a global symmetry. As discussed previously, the low energy worldvolume dynamic on the $1 + 3$ dimensional $D3$ -branes in the presence of $D7$ -branes and an O_7 plane preserve 8 supercharges, classically leading to an $\mathcal{N} = 2$ SYM theory with gauge group $SU(2)$ and global symmetry $SO(8)$ which is exactly Seiberg-Witten theory. The position of the $D3$ -brane along the Coulomb branch is given by the expectation value of the complex scalar in the adjoint representation of the gauge group in the $\mathcal{N} = 2$ $SU(2)$ vectormultiplet:

$$\langle \phi \rangle = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}. \quad (2.106)$$

The u -plane parametrizing the classical moduli space is written as

$$u = \frac{1}{2} \text{Tr } \phi^2 = v^2. \quad (2.107)$$

Similarly, the position of the $D3$ -brane along the Higgs branch - which is the direction along the O_7 and $D7$ -branes, is parametrized by the expectation value of the two complex scalars in the $\mathcal{N} = 2$ hypermultiplet. The classical curve corresponding to the dynamic of the classical $Sp(2)$ gauge group is given by

$$y^2 = (x^2 - u)^2. \quad (2.108)$$

The zeroes of this curve, located at $u = x^2$ indicate the location of the singular region on the classical moduli space where the point of enhanced gauge symmetry is (let $x = 0$). Since the u -parameter corresponds to the location of the $D3$ -brane,

the aforementioned singular point of enhanced gauge symmetry is consistently located where the $D3$ -brane coincides with the O_7 plane at $v = 0$. The classical moduli space can thus be understood as a probe $D3$ -brane in the background of $D7$ -branes and a O_7 -plane. The complex gauge coupling on the $D3$ -brane describing the $\mathcal{N} = 2$ SYM physics is given by the type IIB complex dilaton:

$$\tau = a + \frac{i}{g_s}, \quad (2.109)$$

where a here denotes the axion. Since the $D7$ -branes and O_7 -plane carry charges under the complex dilaton, the presence of these objects in the background of the $D3$ -brane modifies the value of the complex dilaton and consequently, the value of the complex gauge coupling [40]. In particular, when the $D3$ -brane goes once around a $D7$ -brane, the complex gauge coupling picks up a monodromy, transforming as $\tau \rightarrow \tau + 1$. Since there is +1 unit of 7-brane charge where $v = \pm m_i$ and -8 unit of 7-brane charge at $v = 0$, the gauge coupling, far from the point where the $D3$ -brane coincides with either the $D7$ -brane or the Orientifold plane, is given by:

$$\tau(v) = \tau_0 + \frac{1}{2\pi i} \left[\sum_{i=1}^4 (\log(v - m_i) + \log(v + m_i)) - 8 \log v \right]. \quad (2.110)$$

This can also be rewritten in terms of the gauge invariant modulus u (2.107) as:

$$\tau(u) = \tau_0 + \frac{1}{2\pi i} \left[\sum_{i=1}^4 \log(u - m_i^2) - 4 \log u \right], \quad (2.111)$$

(the coefficient 4 in the third term is due to the fact that W^\pm bosons carry twice the electric charges of the quarks). The presence of the logarithmic terms in the equations above indicates that this is a semiclassical result, corresponding to a prepotential at 1-loop. Since $\text{Im}\tau$ is large and negative for small values of u , this expression for the complex gauge coupling can not be exact as it does not satisfy the condition $\text{Im}\tau \geq 0$ everywhere on the moduli space.

When including non-perturbative corrections, the exact effective coupling is a modular parameter $\tau(u)$ of a torus described by the elliptic curve of the form (2.93)

where $u \equiv z$. Under non-perturbative corrections the O_7 splits into 2 (p, q) 7-branes of charge $(1, 0)$ and $(1, -1)$ corresponding respectively to a monopole and a dyon. This is shown in figure (2.3). The (p, q) 7-branes are at a distance of order $\exp(2\pi i\tau_0)$ from the

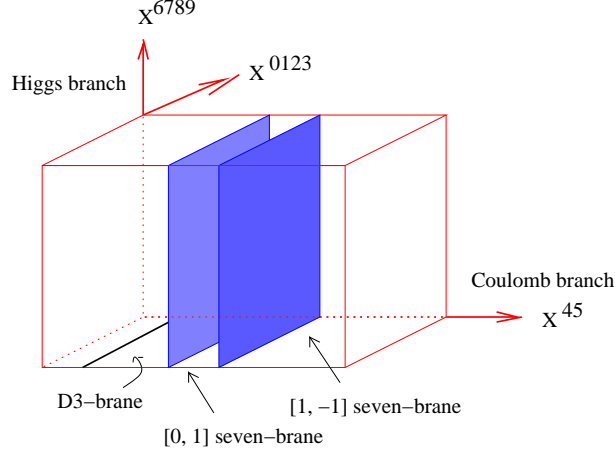


Figure 2.3: The quantum corrections in the $Sp(2)$ theory maps to the splitting of the orientifold plane into two (p, q) seven-branes in F-theory.

original point $u = 0$. The parameter $\exp(i\pi\tau_0) \equiv \Lambda$ is the analogue of the QCD scale of the theory. In the limit $\tau_0 \rightarrow i\infty$ ($g_s \rightarrow 0$) and $|u| \gg |\Lambda|^2$, the two singularities coincide and we recover the semiclassical picture (2.111) [72, 40]. Under the splitting of the orientifold plane, the $D7$ and $D3$ -mirror branes no longer exists. Since there are no longer charged W^\pm bosons in the picture, the gauge group on the $D3$ -brane can no longer enhance and is $U(1)$ everywhere on the moduli space. When the $D3$ -brane coincide with either the “dyonic”, “monopole” or $D7$ -brane, the $(1, -1)$, $(0, 1)$ or $(1, 0)$ string between them becomes arbitrarily light i.e the quark becomes massless, corresponding the massless hypermultiplet in the fundamental representation of the gauge group. The curve capturing the dynamic of the quantum moduli space is:

$$y^2 = (x^2 - u)^2 - \Lambda^4. \quad (2.112)$$

The zeroes located at $u = x^2 - \Lambda^2$ and $u = x^2 + \Lambda^2$ (let $x = 0$) correspond to the two points ($u = \pm\Lambda^2$) where a monopole hypermultiplet and a dyon hypermultiplet become massless respectively. These two points correspond to the positions where the

$D3$ -brane coincide with the two (p, q) 7-branes respectively. The final picture of the quantum moduli space has 1 $D3-2(p, q)$ 7-branes, 4 $D7$ where the $D7$ -branes are free to move in the u -plane but the $(0, 1)$ and $(2, 1)$ are stuck at $u = \pm\Lambda^2$. Therefore, the final quantum picture leading to Seiberg-Witten theory has 6 7-branes in type IIB. Remembering that F-theory is obtained by putting at most 24 7-branes transverse to the type IIB moduli space (u -plane), one sees that the full moduli space of F-theory on $\mathbb{K}_3 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$ with 24 7-branes captures 4 copies of $\mathcal{N} = 2$ Seiberg-Witten theory with $N_f = 4$. In F-theory language, the zeroes of the discriminant of (2.112) correspond to points on the moduli space where the fibered torus T^2 degenerates on the u -plane [72, 11, 40].

2.10 Possible background deformations

We have seen so far that for less than 24 7-branes, one can use type IIB on $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$ to capture the information contained in $\mathcal{N} = 2$ SYM theory in 4 dimensions. On the other hand, if we put 24 7-branes transverse to the type IIB moduli space $\mathbb{R}^2/\mathbb{Z}_2$, the latter compactifies to $\mathbb{CP}^1 \cong T^2/\mathbb{Z}_2$ and we can make use of the IIB/F-theory duality

$$\text{Type IIB: } T^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123} = \text{F-theory: } K_3 \times \mathbb{R}^4 \times R^{0123}, \quad (2.113)$$

where the RR charges of the $D7$ -branes is cancelled by having 1 O_7 -plane per bunch of 4 $D7$ -branes, leading to 4 copies of Seiberg-Witten theory. Recall the direction spanned by certain branched of moduli space of the type IIB theory dual to F-theory:

$$\text{IIB: } \underbrace{\frac{T^2}{\mathbb{Z}_2}}_{\text{Coulomb branch}} \times \underbrace{\mathbb{R}^4}_{\text{Higgs branch}} \times \underbrace{\mathbb{R}^{0123}}_{\text{Minkowski}}, \quad (2.114)$$

staring at this long enough, one realized that certain deformations of this background are allowed and might lead to new and interesting physics. These are summarized in the table below [27]:

From (2.114), we see that each deformation in Table (2.2) corresponds to a different compactification of the Higgs branch. Here, we will discuss each of them briefly.

$IIB : \mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$			
$\mathbb{R}^4 \rightarrow \mathbb{R}^4/\mathbb{Z}_k$	$ALE_{k>1}$	non-compact	$\mathcal{N} = 2$
$\mathbb{R}^4 \rightarrow \mathbb{T}^4/\mathbb{Z}_2$	K3	compact	$\mathcal{N} = 2, 1$
$\mathbb{R}^4 \rightarrow \mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^2$			$\mathcal{N} = 1$

Table 2.2: Possible background deformations of the F-theory geometry

The main content of this thesis will focus exclusively on the first deformation and on the new physics that emerges from it. A detailed analysis of the other cases can be found in [27].

1. Chapter 3 will deal with the first deformation consists in taking:

$$\mathbb{R}^4 \rightarrow \mathbb{R}^4/\mathbb{Z}_k, \quad (2.115)$$

under type IIB/F-theory duality, it leads to:

$$IIB: \mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_k \times \mathbb{R}^{0123} \rightarrow \text{F-theory: } K3 \times \mathbb{R}^4/\mathbb{Z}_k \times \mathbb{R}^{0123}. \quad (2.116)$$

Where $\mathbb{R}^4/\mathbb{Z}_k \cong TN_k$ with TN_k a k -centered Taub-NUT space. The new type IIB/F-theory obtained preserve $\mathcal{N} = 2$ supersymmetry in 4 dimensions. We will analyze the physics of multiple $D3$ -branes probing F-theory on $K3 \times TN_k$ with 7-branes wrapped on the TN_k . We will study this model:

- in the conformal and non-conformal limit
- see how it reproduces Gaiotto-model, Benini-Benvenuti-Tachikawa-model, and Gaiotto duality
- discuss about the supergravity dual of the conformal regime

- see how it leads to $\mathcal{N} = 2$ cascade in the non-conformal limit

2. The second deformation

$$\mathbb{R}^4 \rightarrow \mathbb{T}^4/\mathbb{Z}_2, \quad (2.117)$$

was first studied by [26] where the authors show that it preserved both $\mathcal{N} = 2$ and $\mathcal{N} = 1$ susy. Under type IIB/F-theory duality, they obtained:

$$\text{IIB: } \mathbb{T}^2/\mathbb{Z}_2 \times K3 \times \mathbb{R}^{0123} \rightarrow \text{F-theory: } K3 \times K3 \times \mathbb{R}^{0123}, \quad (2.118)$$

where $\mathbb{T}^4/\mathbb{Z}_2 \cong K3$. [27] revisited this model and studied the following:

- supergravity solutions for $D3 - \bar{D}3$ probing $K3 \times K3$ in F-theory with G -fluxes
- observed duality between abelian instantons and G -fluxes in M-theory
- connected to non-trivial M(atrrix) theory on $K3 \times K3$ with fluxes

3. The last deformation consists in

$$\mathbb{R}^4 \rightarrow \mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^2. \quad (2.119)$$

This model was first studied in [37] where it was shown to lead to $\mathcal{N} = 1$ susy theory in 4-dimensions. The associate background is given by

$$\text{IIB: } \left(\frac{\mathbb{T}^2}{\Omega \cdot (-1)^{F_L} \cdot \mathcal{I}_{45}} \times \frac{\mathbb{T}^2}{\Omega \cdot (-1)^{F_L} \cdot \mathcal{I}_{89}} \right) \times \mathbb{R}^2 \times \mathbb{R}^{0123} \quad (2.120)$$

$$(2.121)$$

$$\text{F-theory: } (\text{CY}_3) \times \mathbb{R}^2 \times \mathbb{R}^{0123}. \quad (2.122)$$

Where the new \mathbb{Z}_2 orbifold was written explicitly so that we don't confused it with the existing orbifold where $\mathcal{I}_{89} : x^{8,9} \rightarrow -x^{8,9}$. When lifting to F-theory, the moduli space of type IIB becomes an elliptically fibered Calabi-Yau 3-fold. In this model, [27] studied the physics of multiple $D3$ -branes probing intersecting

7-branes and O_7 -planes background in type IIB or analogously, F-theory on a CY_3 . They analyzed

- a dual map to the heterotic theory on a non-Kähler $K3$ manifold that is not a conformally Calabi-Yau manifold
- new examples of type IIB and M-theory compactifications on non-Kähler manifolds

As mentioned previously, Chapter 3 will address the first background deformation and explore in depth its consequences. Chapter 4 will try to clarify one of the new physical phenomenon raised in Chapter 3.

Chapter 3

F-theory embedding of new supersymmetric gauge theories

3.1 Introduction

Inspired by S-duality and in particular by Argyres-Seiberg duality, Gaiotto [35] asked if the latter strong-weak duality could hold for generic gauge groups. He explored the strongly coupled limit of various $\mathcal{N} = 2$ superconformal gauge theories in four dimensions and found that an Argyres-Seiberg-like duality does hold for $\mathcal{N} = 2$ SCFT with gauge groups of the form $\prod_{i=1}^n SU(N_i)$ and appropriate amount of global symmetry to make the theory conformal. In particular, he found that in all these conformal cases, a dual weakly coupled theory emerges from the very strongly coupled regions in the moduli space. The weakly coupled gauge theory is coupled to interacting $\mathcal{N} = 2$ SCFT with no exactly marginal deformations called T_N . By rearranging these building blocks, he was able to generate a wide class of new $\mathcal{N} = 2$ generalized quivers. Providing a Seiberg-Witten curve for all these generalized quivers, Gaiotto gave a brane description of these gauge theories as N M5-branes wrapped on a Riemann surface [35]. This brane construction provided him with a recipe for constructing four dimensional gauge theories as a compactification of six dimensional $(2, 0)$ SCFT of A_{N-1} type.

3.2 Multiple $D3$ -branes probing seven-branes on a Taub-NUT background

Based on our understanding of Sen [72] and Banks-Douglas-Seiberg's [11] embedding of Seiberg-Witten theory [68, 69] in F-theory [76], it is natural to ask if it is possible to capture Gaiotto's $\mathcal{N} = 2$ $d = 4$ superconformal gauge theories and their associated dualities [35] in F-theory. The aim is to obtain an even more geometric picture than what M-theory already provided and see if the embedding could shed some light on new physics. We will see in this section that our construction succeeds in doing just that.

This is achieved by first deforming the background of the original type IIB $D3/D7$ system by replacing the \mathbb{R}^4 for a more non-trivial four-dimensional space. The simplest non-compact example is an ALE space $\frac{\mathbb{R}^4}{\mathbb{Z}_2}$, or more locally, a Taub-NUT space- see discussion around (2.115). Asymptotically, $\mathbb{R}^4/\mathbb{Z}_2$ is $\mathbb{R}^3 \times S^1$. If the radius of the circle is of finite size, we call the geometry an ALF (asymptotically locally flat) space whereas it is called an ALE (asymptotically locally Euclidean) space if $R \rightarrow \infty$. Although we will refer to $\mathbb{R}^4/\mathbb{Z}_2$ or more generally to $\mathbb{R}^4/\mathbb{Z}_k$ as ALE space throughout this text, we will always take the constant radius limit of the asymptotic circle of $\mathbb{R}^4/\mathbb{Z}_k$. We thus obtain:

$$\begin{aligned} \text{Type IIB on } & \frac{\mathbf{T}^2}{\Omega \cdot (-1)^{F_L} \cdot \mathcal{I}_{45}} \times \frac{\mathbb{R}^4}{\mathbb{Z}_2} \times \mathbb{R}^{0123} = \\ \text{F Theory on } & \text{K3} \times \frac{\mathbb{R}^4}{\mathbb{Z}_2} \times \mathbb{R}^{0123} \end{aligned} \quad (3.1)$$

probed by a single $D3$ -brane. As we will discuss below, once we increase the number of $D3$ -branes, we can also make the Taub-NUT space multi-centered without breaking further supersymmetries. The brane configuration is given by figure 3.1 and table 3.2. In our configuration the $D3$ -branes are oriented along the spacetime $x^{0,1,2,3}$ directions. The seven-branes (and seven-planes) are parallel to the $D3$ -branes and also wrap multi-centered Taub-NUT space oriented along $x^{6,7,8,9}$. Therefore as before, the Coulomb branch will be the complex $u \equiv x^4 + ix^5$ plane, whereas the Higgs branch will be along the Taub-NUT space. The supersymmetry of this configuration

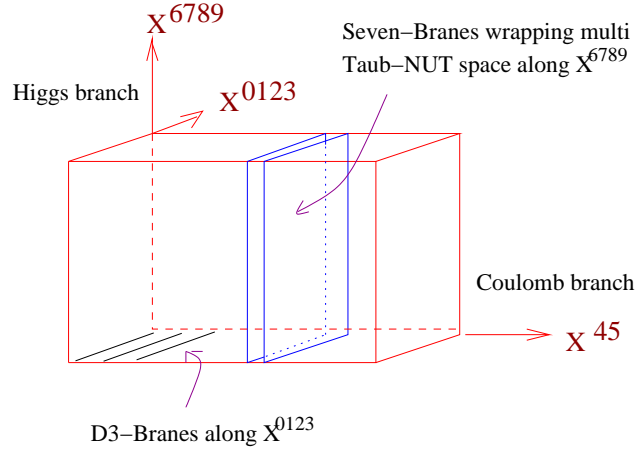


Figure 3.1: Multiple $D3$ -branes probing seven-branes on a multi Taub-NUT geometry.

still remains $\mathcal{N} = 2$ as one can incorporate a Taub-NUT space in a $D3/D7$ system without further breaking susy. A simple trick to prove this statement is to T-dualize our construction¹ along the x^6 direction - since T-duality does not break susy. There, one finds that our brane picture is given by Table 3.2 which the famous Witten's construction [79] preserving eight supercharges.

Type IIB (DSW _{IR})	$\xrightarrow{T_{x^6}}$	Type IIA (Witten)
D3 0123		D4 0123 . . 6 . .
D7 0123 . . 6789		D6 0123 . . . 789
TN 6789		NS_5 012345

Table 3.1: Naive link between our model in the IR and type IIA existing construction

¹ At low energy i.e at far IR-this will be refined latter

3.3 Brane anti-brane on a Taub-NUT background

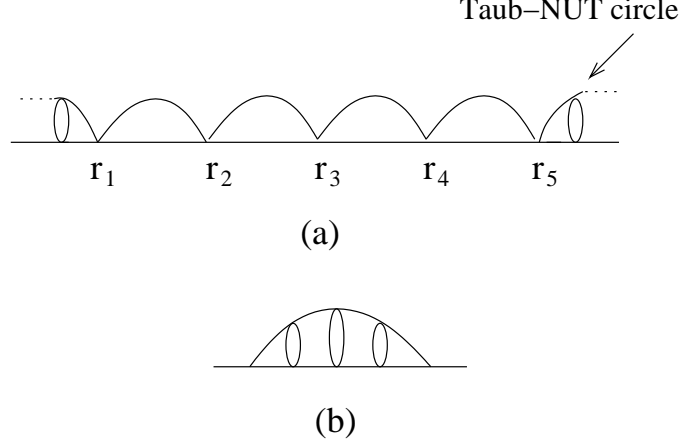


Figure 3.2: In fig (a) the singularities of Taub-NUT space are shown. The compact direction is the Taub-NUT circle that is fibered over the base. The various points at which the circle (which is along x^6 direction in the text) degenerate are the singular points. Between two singular points form a two-cycle, as shown in fig (b), once we assume that the x^6 circle is degenerating along a line parametrized by the x^7 coordinate. Thus the \mathbf{P}^1 's are labelled by $x^{6,7}$ coordinates in the text.

Let us start with a singular \mathbb{Z}_2 ALE space along directions $x^{6,7,8,9}$. The node is really a 5-plane filling the remaining directions. Close to the singular point $x^{6,7,8,9} = 0$, the space can be replaced by a 2-centre (separated in x^6) Taub-NUT metric with coincident (in $x^{0,1,2,3}$) centres see figure (3.2). This is equivalent to saying that we have two coincident Kaluza-Klein monopoles. We also know [8] that the \mathbb{Z}_2 orbifold hides half a unit of B_{NS} flux through the shrunk 2-cycle Σ . The four moduli associated to this ALE space are three geometrical parameters, which can be thought of as the blowup of the ALE to form a smooth Eguchi-Hanson metric, and the B_{NS} flux [8].

Take a $D3$ -brane transverse to the ALE space, filling the directions $x^{0,1,2,3}$. (More generally we start with r such $D3$ -branes.) When the ALE space is singular, the world-volume theory of the 3-brane has two branches: a Higgs branch, when the brane is separated from the singularity along $x^{6,7,8,9}$, and a Coulomb branch when the brane hits the singularity and dissociates into a pair of fractional branes which can

move around only in the $x^{4,5}$ directions. However, if the ALE space is blown up, then the Coulomb branch gets disconnected from the Higgs branch because the 3-brane cannot dissociate supersymmetrically into pair of fractional branes.

The fractional $D3$ -brane is interpreted as D5-brane wrapped on a \mathbf{P}^1 with fluxes or $\overline{D5}$ -brane wrapped on a \mathbf{P}^1 with different choice of fluxes (see details below). Therefore an integer $D3$ -brane would be a pair of five-branes whose D5-brane charges cancel, hence they are really a $D5$ - $\overline{D5}$ (D5-brane – anti-D5-brane) pair (see also [25], [67, 45, 2] where a somewhat similar model has been discussed). However, they carry $D3$ -brane charge proportional to the relative difference of five-brane fluxes by virtue of the Chern-Simons coupling on D5-branes. Denoting the world-volume gauge field strength on the D5-brane by F_1 , we have the coupling

$$\int (B_{\text{NS}} - F_1) \wedge C_4 \quad (3.2)$$

where C_4 is the self-dual 4-form potential in the type IIB string. At the orbifold point we have $\int_{\Sigma} B_{\text{NS}} = \frac{1}{2}$ and hence half a unit of $D3$ -brane charge. The $\overline{D5}$ (anti-D5-brane) (whose world-volume gauge field strength is denoted by F_2) will have a coupling

$$- \int (B_{\text{NS}} - F_2) \wedge C_4. \quad (3.3)$$

Now let us also turn on a world-volume gauge field strength F_2 on the anti-D5-brane and give it a flux of +1 unit through the vanishing 2-cycle Σ (more generally, we assign unit flux to the relative gauge field $F_- = F_2 - F_1$). In this configuration, the $D5$ - $\overline{D5}$ pair has in total $D3$ -brane charge equal to 1, or more generally:

$$\int (F_2 - F_1) \wedge C_4 \equiv \int_{\Sigma \times \mathbb{R}^{0123}} F_- \wedge C_4. \quad (3.4)$$

In a slightly more generalized setting with multi Taub-NUT space the situation is somewhat similar. To see this, let us consider a Taub-NUT space with m singularities as shown in figure 3.2. Once we bring the $D3$ -branes near the Taub-NUT singularities,

they decompose as m copies of $D5-\overline{D5}$ wrapping the various two-cycles of the Taub-NUT space. Each of the wrapped k 'th $D5$'s can be assumed to create a fractional $D3$ -brane on its world-volume via the world-volume $F_{1,k}$ fluxes by normalising the total integral of F_1 over all the two-cycles to equal the number of integer $D3$ -branes. The $\overline{D5}$ branes, on the other hand, are used only to cancel the $D5$ charges as their world-volume fluxes are taken to be zero. These fractional $D3$ -branes can now move along the Coulomb branch as expected. T-dualising this configuration gives us $D4$ -branes between the $NS5$ -branes which may be broken and moved along the Coulomb branch, see table (3.2). The above way of understanding the fractional branes has two immediate advantages:

- Since every \mathbf{P}^1 of the multi Taub-NUT space is wrapped by $D5-\overline{D5}$, and the system is symmetrical, one is restricted to switching on same gauge fluxes on each of the \mathbf{P}^1 's. However for non-compact \mathbf{P}^1 's this restriction doesn't hold as the wrapped branes give rise to flavors and not colors, and one may switch on different fluxes. This will be used to understand the Hanany-Witten brane creation process later in the text.
- If we are interested in non-conformal scenarios ($\beta = N_f - 2N_c \neq 0$), one way to break the balance between the number of color branes N_c and the number of flavor branes N_f is to manually wrap *additional* $D5$ -branes on each of the \mathbf{P}^1 . Since for the conformal case, the number of $D5-\overline{D5}$ on each of the \mathbf{P}^1 's are the same, the presence of additional $D5$ -branes will break conformal invariance leading to cascading theories².

In the above analysis, we assumed that quantum corrections were taken into account and that the O_7 -planes dissociated into (p, q) 7-branes. Henceforth we will assume that the F-theory background probed by the r $D3$ -branes is described completely in

² Instead, if we wrap additional $\overline{D5}$'s on the \mathbf{P}^1 's, they will break supersymmetry. Also the number of $D5-\overline{D5}$'s on each \mathbf{P}^1 should remain same so as to cancel the tachyons across each wrapped \mathbf{P}^1 's.

terms of the seven-branes wrapped on multi Taub-NUT space which, alternatively, would also mean that the moduli space is a \mathbf{P}^1 with 24 transverse seven-branes (not to be confused with the P^1 of TN_k), i.e:

$$\frac{\mathbf{T}^2}{\Omega \cdot (-1)^{F_L} \cdot \mathcal{I}_{45}} \longrightarrow \mathbf{P}^1. \quad (3.5)$$

Thus going to the Coulomb branch, by tuning all of $x^{6,7,8,9}$ to 0, the picture is somewhat different. At this point a $D3$ -brane splits into a pair of fractional branes which can move independently along $x^{4,5}$. On the T-dual side, one understands this as regular $D4$ -branes, stretched along 2 NS5-branes on a compactified x_6 direction. The $D4$ -branes split into two partially wrapped pieces on each side of the x^6 circle (now fractional $D4$ -branes), moving independently along the Coulomb branch $x^{4,5}$ i.e. along the two NS5-branes. Of course, this is simply the classical picture, in the correct non-perturbative scenario (reproducing our F-theory picture) the $D4$ -branes and NS5-branes form thin M5-branes tubes. .

To summarize, the type IIB picture on the Coulomb branch is that the relative world-volume gauge field strength F_- on the $D5\text{-}\overline{D5}$ pair must be turned on over the 2-cycle Σ and gives rise to a 3-brane in the space transverse to that cycle. The spacetime B_{NS} flux over Σ changes the relative tensions of the wrapped $D5$ -brane and anti- $D5$ -brane keeping the total constant. The directions of various branes and fluxes in our set-up therefore is given in table 3.2. Supersymmetry is preserved because the $D5\text{-}\overline{D5}$ pairs wrap vanishing 2-cycles of the multi Taub-NUT space, in addition to the conditions mentioned earlier. There are also additional background fluxes like axion-dilaton, two-forms and four-forms. The metric of the Taub-NUT space will be deformed due to the backreactions of the branes and fluxes, that we will discuss later. All these effects conspire together to preserve $\mathcal{N} = 2$ supersymmetry on the fractional probe $D3$ -branes. Note that if the Taub-NUT cycles are blown-up then, in the presence of the seven-branes, supersymmetry will be broken.

Directions	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—	*	*	—	—	*	*
$\overline{\text{D5}}$	—	—	—	—	*	*	—	—	*	*
D7	—	—	—	—	*	*	—	—	—	—
Taub-NUT	*	*	*	*	*	*	—	—	—	—
Fluxes	*	*	*	*	*	*	—	—	*	*

Table 3.2: The orientations of various branes and fluxes in our set-up. The dashed lines for the branes are the directions parallel to the world-volume of the branes; and for the fluxes and Taub-NUT space are the directions along which we have non-trivial fluxes and metric respectively. The stars denote orthogonal spaces. Supersymmetry is preserved in the presence of vanishing cycles and background fluxes.

3.4 Anomaly inflow, anti-GSO projection and brane transmutation

There is an interesting subtle phenomenon that happens to our system when we switch on a time-varying vector potential $A_\mu(t)$ along the Taub-NUT space. However before we go about discussing this in detail, we want to point out an important property of the underlying Taub-NUT space, namely, the existence of a normalizable harmonic two form Ω . For m -centered Taub-NUT there would be equivalently m normalizable harmonic forms $\Omega_i, i = 1, 2, \dots, m$. The existence of these harmonic forms are crucial in analyzing the phenomena that we want to discuss.

To see what happens when we switch on time-varying Wilson line, note first that the seven-brane wrapping the Taub-NUT space will give rise to a $D3$ -brane bound to it. The charge of the $D3$ -brane is given by the non-trivial B_{NS} background on the Taub-NUT. To see this, consider some of the couplings on the world volume of the $D7$ -brane (we are neglecting constant factors in front of each terms):

$$\int *C_0 + \int C_4 \wedge F \wedge B_{\text{NS}} + \int C_4 \wedge F \wedge F + \dots \quad (3.6)$$

These couplings are derived from the Wess-Zumino coupling $\int C \wedge e^{B-F}$, where C is the formal sum of the RR potentials. The first term $\int *C_0$ gives the charge of the $D7$ -brane.

Coming back to our phenomena, notice that we cannot turn on a flat connection on this space. Instead, a self-dual connection can be turned on. This self-dual connection is of the form:

$$F = dA = \Omega, \quad (3.7)$$

where Ω is the unique normalizable harmonic two-form on the Taub-NUT space. This harmonic two form, being normalizable, goes to zero at infinity, hence we have a flat connection there. Recall that at infinity, the multi-centered Taub-NUT space asymptotes to $\mathbb{R}^3 \times S^1$ therefore, at infinity, the flat connection corresponds to a Wilson line on that S^1 .

The above choice of background (3.7) however doesn't take the fluctuations of gauge fields into account. A more appropriate choice for our case is to decompose the field strength F as

$$F = \Omega + F_1. \quad (3.8)$$

instead of just (3.7). Now F_1 will appear as a gauge field on the $D7$ (or Taub-NUT plane). Inserting (3.8) in (3.6) and integrating out Ω , we get the required $D3$ -brane charge (see also [25] for more details). This confirms that a bound state of a $D3$ with the $D7$ -brane appears once we switch on a self-dual connection (which is of course the Wilson line for our case).

However the situation at hand demands a *time-varying* gauge field on the world volume of the $D7$ -brane. A typical time-varying gauge field A_μ can be constructed from F_1 in (3.8) by making it time-dependent. Such a time-varying gauge field creates a chiral anomaly along the S^1 at the asymptotic region of the Taub-NUT space. This $1 + 1$ dimensional anomaly is of the form [9]

$$\int d^2x \, \omega \epsilon^{ab} \partial_a A_b, \quad (3.9)$$

where ω is the gauge transformation parameter. Another way to see this anomaly is to dualize the $D7$ -brane and the Taub-NUT space into a $D6/D4$ system oriented along $x^{0,1,2,3,4,5,6}$ and $x^{0,6,7,8,9}$ respectively³. The chiral anomaly is along the x^6 direction.

It is suggested in [25] that the term cancelling the aforementioned anomaly is given by:

$$S = \int G_5 \wedge A \wedge F \quad (3.10)$$

on the world volume of the seven-brane. Here $G_5 = dC_4$, the pullback of the background four-form, in the absence of any source. The cancellation takes place via anomaly inflow. We have a coupling, (3.10), in $(7+1)d$ spacetime. Along a $(1+1)d$ subspace of this, chiral fermions propagate and give rise to the anomaly (3.9). Since $D3$ -brane is the source for G_5 , we find that changing the Wilson line produces a change of flux of G_5 . In other words, a gauge transformation $\delta A = d\omega$ on the world-volume will vary (3.10) by:

$$- \int dG_5 \wedge (\omega F) \quad (3.11)$$

Since $dG_5 \neq 0$ in the presence of a source of G_5 flux⁴, we end up with:

$$\delta S = - \int d^2x \, \omega \epsilon^{ab} \partial_a A_b \quad (3.12)$$

resulting in the inflow which cancels the anomaly (3.9) by creating a $D3$ -brane.

The story is however not complete. There is an *additional* phenomena that happens simultaneously that actually reduces the number of $D3$ -branes instead of increasing it (as we might have expected from the above discussion). This additional phenomena relies on the dissociation of the $D3$ -branes into $D5\text{-}\overline{D5}$ pairs discussed

³ Use the following set of dualities to go from one picture to another: T-dualities along $x^{6,1,2,3}$ then a S-duality followed by another T-duality along x^6 .

⁴ Since G_5 is self-dual, this switches on a $D3$ -brane with orientations along $x^{0,1,2,3}$ directions.

in the previous subsection. Recall that the tachyon between the D5 and the $\overline{\text{D5}}$ is cancelled for $F_- \equiv F_2 - F_1 = \pm 1$. Here we set

$$F_2 = 1, \quad F_1 = 0, \quad (3.13)$$

which also implies that $F_- = 1$ in (3.4), giving rise to a unit $D3$ -brane charge.

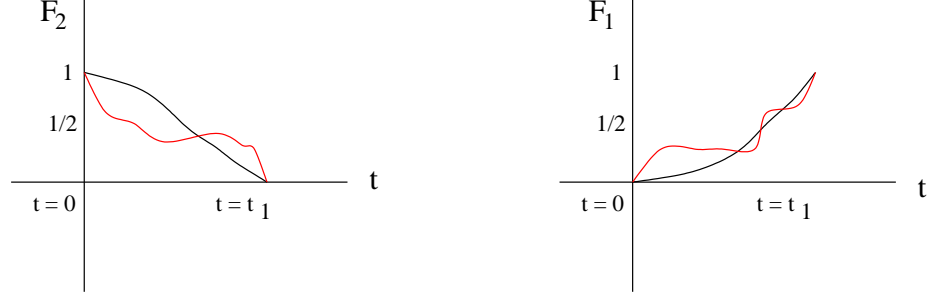


Figure 3.3: The gauge fluxes on the D5 and the $\overline{\text{D5}}$ branes are changed with time. The red and the black lines denote two different changes with the same initial and final values.

Now imagine that we change the world volume fluxes as described in figure 3.3, starting with (3.13):

$$t = 0 \quad F_1 = 0, \quad F_2 = 1$$

$$t = t' \quad F_1 = \frac{1}{2}, \quad F_2 = \frac{1}{2}$$

$$t = t_1 \quad F_1 = 1, \quad F_2 = 0$$

Table 3.3: Variation of world-volume fluxes with time

where $(t = 0) < t' < (t = 1)$. Since the $D3$ -brane charge is given by (3.4), we see that at $t = t'$, the $D3$ -brane charge vanishes. On the other hand, when $t = t_1$, the relative difference in the fluxes $F_- = F_2 - F_1$ is negative, indicating that the $\text{D5-}\overline{\text{D5}}$ are generating an $\overline{\text{D3}}$. Note that since the norm of F_- is still equal to 1, the tachyon between the D5 and the $\overline{\text{D5}}$ is still massless and the supersymmetry is not broken. We have just seen that variations in the flux of $\text{D5-}\overline{\text{D5}}$ transmute a $D3$ -branes into a $\overline{\text{D3}}$ -brane. Thus switching on a time-varying Wilson line has the following two effects:

- Chiral anomaly cancellation via anomaly inflow and creation of a new $D3$ -brane.
- $D3$ -brane transmutation to an $\overline{D3}$ brane via flux change.

Together these two effects would remove one of the existing $D3$ -brane in the system. Therefore the color degree of freedom would change via this process. If we do this multiple times, we can reduce the number of $D3$ -branes in the model. One way I like to think about the above two phenomena is in the brane language of Hanany-Witten [50]. It was shown in [9] that turning on a time varying gauge field $A_1(t)$ on intersecting D5-branes along (12345) and (16789) is T-dual to the relative motion of orthogonal D4-branes along the transverse direction $x^1(t)$, creating a fundamental string every time the fourbranes cross each other. [9] showed that their setup is T-dual to that of Hanany-Witten's. Since we know that our brane construction is also T-dual to Hanany-Witten's model- see table 3.2, one can think of the chiral anomaly cancellation via anomaly inflow and creation of a $D3$ -brane simply as brane creation in Hanany-Witten figure 3.4. If we combine this phenomenon with flux change which

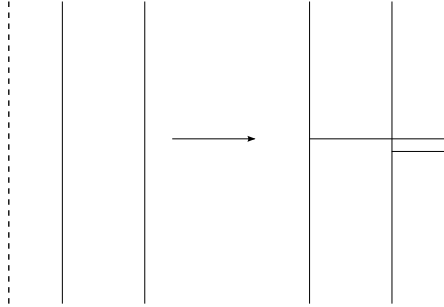


Figure 3.4: HW brane creation: solid vertical lines are NS5-branes (23456), dashed vertical line is D5-brane (56789), solid horizontal line is a $D3$ -brane along (156) .

transmute a $D3$ -brane into an anti- $D3$ brane, we obtain brane annihilation as shown in figure (3.5). We will say a few more words on the Hanany-Witten mechanism when we will discuss about the brane networks. Of course in the *absence* of the Taub-NUT space, none of the above arguments would work, and so there would be no brane creation/annihilation. This is perfectly consistent with our expectation.

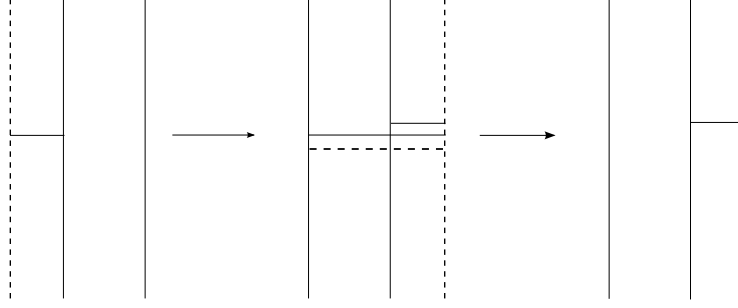


Figure 3.5: HW brane creation: solid vertical lines are NS5-branes (023456), dashed vertical line is D5-brane (056789), solid horizontal line is a $D3$ -brane along (0156), horizontal dashed line is an anti- $D3$ -brane (0156)

Coming back to the system under study, imagine now that we switch on gauge fluxes $F = n\Omega$. This would imply that we have two new sources of the form:

$$\frac{n^2}{2} \int C_4, \quad \text{and} \quad n \int C_4 \wedge (B_{\text{NS}} - F_1). \quad (3.14)$$

The latter doesn't break supersymmetry as was explained in [25]. In fact overall the supersymmetry will never be broken if we take r $D3$ -branes and we simultaneously consider m -centered Taub-NUT. Therefore the r $D3$ -branes wrap m different vanishing 2-cycles.

From the above discussions we see that we have two models that are *dual* to each other while preserving supersymmetry. The duality criteria for our case can be presented in the following way:

- r $D3$ -branes probing seven-branes wrapping a m -centered Taub-NUT space. The $D3$ -branes dissociate as m copies of $D5\text{-}\overline{D5}$ pairs that move along the Coulomb branch as depicted in figure 3.8. The seven-branes could be arranged to allow for any global symmetries and axion-dilaton moduli, including the conformal cases.
- Time-varying self-dual connections on the seven-branes along the Taub-NUT direction that create *and* transmute $D5\text{-}\overline{D5}$ sources by changing the F_i in (3.2) and (3.3). Due to this some $D3$ -branes may annihilate, thereby changing the local and possibly the global symmetries of the model.

Our claim therefore is the following. The above two dual descriptions, coming from chiral anomaly cancellation, $D3$ -brane creation and $D3$ -brane transmutation, are related by the recently proposed Gaiotto dualities. In the next subsection we will supply more evidences for this conjecture.

Note that the total moduli in both the models are exactly similar, although both color and flavor degrees of freedom may apparently differ. The seven-branes could be arranged such that we could either have F-theory at constant couplings *a la* [22], or non-constant couplings. However due to the underlying F-theory constraints, the flavor degrees of freedom remain below 24 although the color degrees of freedom could be anything arbitrary. In addition to that there is also a M-theory uplift of our model that is quite different from the M-theory brane constructions studied by Witten [79] and Gaiotto [35]. We will discuss this soon.

3.5 Mapping to Gaiotto theories and beyond

Now that we deformed the background and presented our model, let us try to reproduce some of the Gaiotto gauge theories construction [35]. Our first map will be to the brane network model studied by [14] recently. We will then argue how gravitational duals for our models, at least in the conformal limit, may be derived. These gravity solutions should be compared to the recently proposed gravity duals given in [36]. We will see that our model can be extended to the non-conformal cases in two ways: by moving the seven-branes around or by wrapping by hand fractional fivebranes. In fact we will see an interesting class of *cascading* $\mathcal{N} = 2$ models appearing naturally out of our constructions. Additionally, new states in the theory could appear in the generic cases when the $D3$ -brane probes are connected by string junctions or string networks. In the later part of this section we will give some details on these issues, extending the scenario further.

3.5.1 Mapping to the type IIB brane network models

Recently the authors of [14] have given a set of interesting brane network models that may explain certain conformal constructions of the Gaiotto models, including ways to see how the Gaiotto dualities occur from the networks. The obvious question

now is whether there exist some regime of parameters in our set-up that could capture the brane network models of [14].

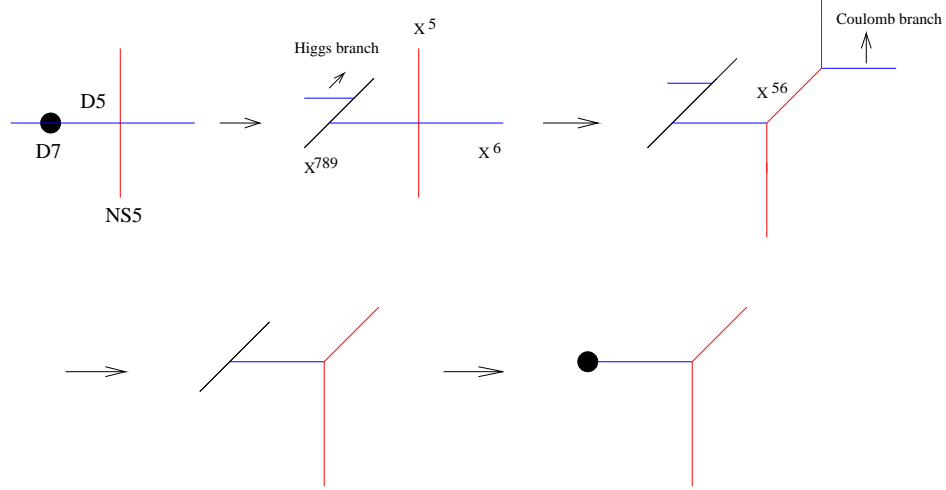


Figure 3.6: The simplest brane junction from our Taub-NUT configuration.

It turns out the mapping to [14] is not straightforward. The orientations of various branes in our set-up are given in table 3.4. A naive T-duality along x^4 and x^6 direction will convert the Taub-NUT space to a NS5-brane oriented along $x^{0,1,2,3,4,5}$ and the $D7$ -brane into another $D7$ -brane oriented along $x^{0,1,2,3,4,7,8,9}$. However the $D5\text{-}\overline{D5}$ pairs will continue as $D5\text{-}\overline{D5}$ pairs although with a slightly different orientation.

Directions	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—	*	*	—	—	*	*
$\overline{D5}$	—	—	—	—	*	*	—	—	*	*
$D7$	—	—	—	—	*	*	—	—	—	—
Taub-NUT	*	*	*	*	*	*	—	—	—	—

Table 3.4: The orientations of various branes in our set-up. Same as the earlier table but now the flux informations are not shown.

This is not what we would have expected for the model of [14]. Furthermore, because of the fluxes as well as other fields that have non-trivial dependences along the $x^{4,6}$ directions T-dualities along these directions are not possible. Therefore there is no simple map to the brane network model of [14]. However we can go to a corner of the moduli space of solutions where:

- (a) We are at low energies i.e at far IR, so the $D5-\overline{D5}$ pairs behave as fractional $D3$ -branes,
- (b) We have delocalized completely along the two T-duality directions $x^{6,4}$.

Under these two special cases together, we can T-dualize along $x^{6,4}$ directions to convert our configuration to the brane network model of [14] as depicted in figures 3.6 and 3.7. In figure 3.6 the configuration in table 3.4 is T-dualized following the above criteria to get to the brane intersection model in the top left of the figure. Motion in the Coulomb branch is precisely the decomposition of the $D3$ -brane into $D5-\overline{D5}$ pair, such that each of them support a fractional $D3$ -brane. Once we have the fractional $D3$ -branes we can move one of them along the $x^{4,5}$ direction⁵. On the other hand we also need to break the $D5$ -brane on the seven-brane and move this along the Higgs branch, as depicted in figure 3.6. This is achieved by expressing the fractional $D3$ -brane (on the $D5$ -brane) as an *instanton* on the seven-brane and then further decomposing the instanton as fractional instantons on the seven-brane. Moving one set of fractional branes along the Higgs branch will eventually give us the brane junction studied by [14] as shown in figure 3.6. [54] Clearly this T-dual mapping works most efficiently with fractional $D3$ -branes and ignoring their $D5-\overline{D5}$ origins. As we saw before, this dissociation is crucial in the presence of multi Taub-NUT space and therefore the mapping to [14] only works under special circumstances. It also means that once we map our model to [14] we may lose many informations of our model. In particular all the high energy informations, like the presence of $D5-\overline{D5}$ pairs, fluxes and massless tachyons are completely lost on the other side. But certain low energy informations do map from our model to [14]. For example a crucial ingredient of [14] is the Hanany-Witten brane creation process that occurs when we move the $D7$ -brane across the $NS5$ -brane. The $D7$ -brane is located at $x_{(1)}^6$ and the

⁵ Of course this is the generic case. But for wrapped $D5-\overline{D5}$ -branes there could be situations where in the T-dual set-up the $D5$ -brane may terminate on $NS5$ -brane (much like the one in [25]).

NS5-brane is located at $x_{(2)}^6$. The relative motion of the $D7$ -brane will induce following T-duality map:

$$\int d^2y \left[\frac{\partial x_{(1)}^6}{\partial t} - \frac{\partial x_{(2)}^6}{\partial t} \right] \longrightarrow \int d^2y \epsilon^{06} \partial_0 A_6, \quad (3.15)$$

which is of course one term of the chiral anomaly $\int \omega \epsilon^{ab} \partial_a A_b$ as we saw before. A cancellation of the chiral anomaly therefore maps to the brane creation picture of [14], although the brane transmutation in our model (that relies on the dissociation of $D3$ -brane into $D5$ - $\overline{D5}$ pair) cannot be seen directly from the T-dual model (although there may exist some equivalent picture).

Another interesting ingredient of [14] is the so-called s-rule that preserves supersymmetry. In this configuration the $D5$ -branes ending on same $D7$ -branes must end on *different* NS5-branes, i.e not more than one $D5$ -brane may end on a given pair of NS5-brane and $D7$ -brane, otherwise supersymmetry will be broken. At low energy we saw that T-duality can map our model to [14]. The m -centered Taub-NUT space can map to the multiple configuration of the NS5-branes. Similarly, (p, q) five branes can be understood as explained above. The $D5$ - $\overline{D5}$ pairs wrap the vanishing cycles of the multi Taub-NUT geometry and we may keep r pairs of $D5$ - $\overline{D5}$ with m -centered Taub-NUT space. This means that there may not be a simple map of the s-rule of [14] to our set-up. This is understandable because making a single T-duality to type IIA along x^6 , and removing the seven-branes, give us the NS5/D4 configuration where multiple $D4$ -branes can end on NS5-branes. However in this mapping all information of the *non-local* seven-branes are completely lost including informations about exceptional global symmetries etc. Thus our F-theory model including number of seven-branes captures additional information.

3.5.2 The UV/IR picture and gravity duals

In the limit when we take the number of $D5$ - $\overline{D5}$ pairs to be very large, we expect the near horizon geometry to give us the gravity duals of the associated theories. One may arrange the seven-branes in such a way that the axion-dilaton coupling doesn't run. In that case the corresponding theories should be conformal at least both at

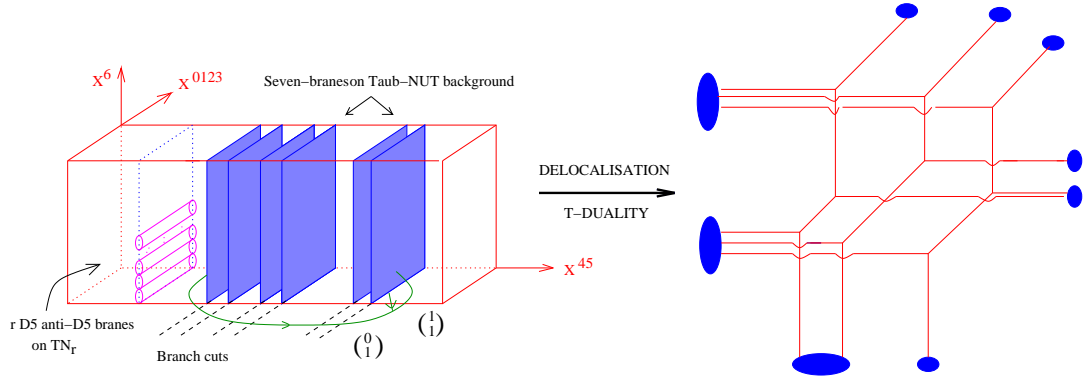


Figure 3.7: Under special arrangement of the seven-branes, delocalization and T-dualities map our model to the brane network studied in [14]. For other configurations there are no simple map to the brane networks. The blue patches on both sides represent the seven-branes. The r $D5-\overline{D5}$ pairs are wrapped on vanishing 2-cycles of a multi Taub-NUT space with m centers.

UV and IR. Recently Gaiotto and Maldacena [36] have studied the gravity duals of some of the Gaiotto models and have provided explicit expressions for the IR pictures. In this subsection we will provide some discussions on this using our set-up. More detailed derivations will be provided in the sequel to this paper.

One aspect of the gravity dual should be clear from the F-theory model that we present here: the UV of the theory should be different from the IR. In fact we expect the UV to be a six-dimensional theory whereas the IR should be four-dimensional. This is borne out from the following observations. In the UV, when the system is probed using high energy wavelengths, the complete large m Taub-NUT singularities should be visible. Therefore UV description should be given by a large r $D5-\overline{D5}$ pairs wrapped on the vanishing 2-cycles of the multi Taub-NUT space. Because of the presence of $D5-\overline{D5}$ pairs we expect the theory should become six-dimensional, and by arranging the seven-branes appropriately the UV should be a 6d SCFT.

On the other hand the IR is simple. Since at IR we are probing the geometry with large wavelengths, the subtleties of the geometry will be completely washed out and we will see only a simple Taub-NUT space with no non-trivial cycles. The $D5-\overline{D5}$ pairs

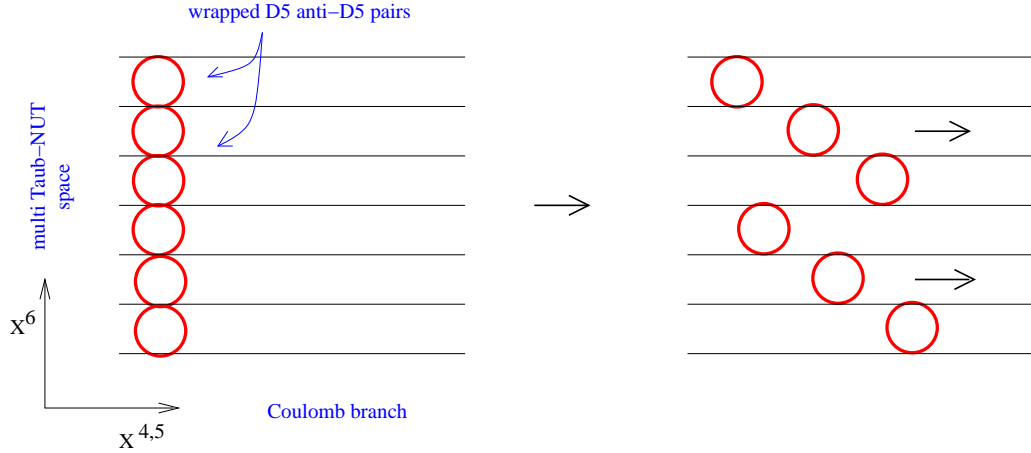


Figure 3.8: D5- $\overline{D5}$ branes wrapped on 2-cycles of a multi Taub-NUT space. When each of these N set of D5- $\overline{D5}$ pairs wrap vanishing cycles of the multi Taub-NUT space, they can give rise to $[SU(N)]^m$ gauge groups where m denote the number of vanishing Taub-NUT 2-cycles. In the next figure various such pairs are broken and moved along the Coulomb branch.

wrapping this geometry will effectively behave as four-dimensional, and therefore the IR geometry should be four-dimensional.

We can make this a bit more precise. The supergravity solution for pairs of D5- $\overline{D5}$ branes on a *flat* background can be given in the following way⁶ [58, 32, 10]:

$$ds^2 = -V_1^{-1}V_2^{-1/2}(dx_0 - kdx_7)^2 + V_2^{-1/2}dx_2^2 + V_2^{-1/2}dx_1^2 + V_2^{-1/2}dx_3^2 \\ + V_2^{1/2}(V_1^{-1}dx_6^2 + dx_7^2) + V_2^{1/2}(dx_4^2 + dx_5^2 + dx_8^2 + dx_9^2), \quad (3.16)$$

where stability and supersymmetry requires us to switch on an electric field F_{0i} with $|F_{0i}|^2 < 1$ and a magnetic field F_{67} with opposite signs on the D5- $\overline{D5}$ pairs. This is slightly different choice of the world-volume fluxes compared to the ones that we took

⁶ It would be interesting to compare the analysis below with the one done in [45] where supergravity solution related to $D7$ and fractional $D3$ -branes is studied. Our analysis is very different from the one in [45] as we will be studying the system from its D5- $\overline{D5}$ perspective, and not from its $D3$ perspective, so as to capture the UV and IR behaviors.

in the previous subsections and in appendix E of [27]. One may however easily verify that both the choices result in identical physics⁷.

Due to the existence of an electric field (with say $i = 6$) there would be bound fundamental strings, and due to the magnetic fields F_{67} there would be bound $D3$ -branes. The $D3$ -brane charge is then typically given by $F_{67}^{(2)} - F_{67}^{(1)}$ as we saw before. If we keep the five-branes at the same point in the $u = x^4 + ix^5$ plane but separate the anti five-branes very slightly along the $r = \sqrt{(x^8)^2 + (x^9)^2}$ directions, then

$$V_i = 1 + \alpha_i \left[\frac{1}{|u|^2 + r^2} + \frac{1}{|u|^2 + (r - \epsilon)^2} \right], \quad k = \beta \left[\frac{1}{|u|^2 + r^2} - \frac{1}{|u|^2 + (r - \epsilon)^2} \right], \quad (3.17)$$

where $\alpha_{1,2}, \beta$ are functions of g_s, l_s ; and the number of fundamental strings, $D3$ -branes and five-branes respectively. Note that in k the two terms come with a relative minus sign, so that when $\epsilon \rightarrow 0$, k vanishes.

The above picture is not complete as we haven't yet accounted for the multi Taub-NUT space and seven-branes. Let us first consider the multi Taub-NUT space oriented along $x^{6,7,8,9}$ directions. The multi Taub-NUT space modifies the $x^{6,7,8,9}$ directions in the following way:

$$ds_{\text{TN}}^2 = \left(1 + \sum_{\sigma} \frac{1}{|\vec{w} - \vec{w}_{\sigma}|} \right) d\vec{w}^2 + \left(1 + \sum_{\sigma} \frac{1}{|\vec{w} - \vec{w}_{\sigma}|} \right)^{-1} \left(dx^6 + \sum_{\sigma} F_i^{\sigma} dx^i \right)^2, \quad (3.18)$$

⁷ In this framework one might worry about the fundamental string oriented parallel to the seven-branes i.e along x^6 direction. This can be dissolved in one of the $D7$ -brane and then moved away in the u -plane, so that local $\mathcal{N} = 2$ supersymmetry remains unaffected. This is equivalent to the statement that we can go to a frame where only world-volume magnetic field is turned on and the electric field is zero. In the framework studied in the earlier sub-section there are only fractional $D3$ -branes and no fundamental strings.

where σ denotes Taub-NUT singularities and $|\vec{w}| = \sqrt{|r|^2 + (x^7)^2}$ denotes the distance along the Taub-NUT space.

In addition to the Taub-NUT space, we also have the seven-branes distributed in some way to give rise to the global symmetries in the theory. For generic distribution of the seven-branes the resulting gauge theory is not conformal. The metric orthogonal to the seven-branes along the u -plane is given by the following expression (see for example [47]):

$$ds_u^2 = \tau_2(u) \left| \eta^2(\tau(u)) \prod_{i=1}^{24} \frac{du}{(u - u_i)^{1/12}} \right|^2, \quad (3.19)$$

where $\tau_2(u)$ is the imaginary part of $\tau(u)$ on the u -plane and $\eta(\tau)$ is the η -function (we are using the notations of [47]).

Now combining (3.19), (3.18) and (3.16) we obtain the total background metric. The dx^6 component of (3.16) should be replaced by the $U(1)$ fibration metric of (3.18) and the (dx^4, dx^5) part of (3.16) should be replaced by the backreaction from the seven-branes, i.e (3.19). Together, the final picture would be pretty involved, and will take the following form:

$$\begin{aligned} ds^2 = & -f_1 V_1^{-1} V_2^{-1/2} (dx_0 - k dx_7)^2 + f_2 V_2^{-1/2} dx_2^2 + f_3 V_2^{-1/2} dx_1^2 + f_4 V_2^{-1/2} dx_3^2 \\ & + f_5 V_1^{-1} V_2^{1/2} \left(dx^6 + \sum_{\sigma} F_i^{\sigma} dx^i \right)^2 + f_6 V_2^{1/2} \tau_2(u) \left| \eta^2(\tau(u)) \prod_{i=1}^{24} \frac{du}{(u - u_i)^{1/12}} \right|^2 \\ & + f_7 V_2^{1/2} dx_7^2 + f_8 V_2^{1/2} (dx_8^2 + dx_9^2), \end{aligned} \quad (3.20)$$

where we expect (f_5, f_7, f_8) to be functions of $(|u|, \vec{w})$ so that informations about the multi Taub-NUT space can be captured⁸. The other f_i would definitely be functions of $|u|$ but could have dependences on other coordinates too. The V_i 's now specify

⁸ Note that the multi Taub-NUT geometry is deformed due to the backreactions of branes and fluxes in the background.

the harmonic functions for all the wrapped D5- $\overline{\text{D5}}$ pairs on the multi Taub-NUT two-cycles.

In the above metric we can go to either the conformal or the non-conformal limits. The conformal limits will be given by some special re-arrangements of the 24 seven-branes whose individual contributions appear in (3.20). The non-conformal limits are of course any generic distributions of the seven-branes in (3.20). In addition, this limit can be probed by adding fractional D5-branes as will be explained in the next section. Each of the two limits would also have their individual UV and IR behaviors. The IR behavior for both the conformal as well as the non-conformal limits shouldn't be too difficult to determine from the above form of the metric (3.20).

At IR we expect that all informations about the multi Taub-NUT space will be washed out (because we are probing the system with wavelengths larger than the resolutions of the Taub-NUT singularities). This means at IR the system is probed by $D3$ -branes. We also expect

$$k \equiv \beta \left[\frac{1}{|u|^2 + r^2} - \frac{1}{|u|^2 + (r - \epsilon)^2} \right] = 0 \quad (3.21)$$

in (3.20), as the D5- $\overline{\text{D5}}$ pairs would effectively overlap, and so there would be no $dx_0 dx_1$ cross-terms in the metric. The metric then takes the following form:

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{V_2}} \left(-f_1 V_1^{-1} dx_0^2 + f_3 dx_1^2 + f_2 dx_2^2 + f_4 dx_3^2 \right) \\ &+ \sqrt{V_2} \left[f_5 V_1^{-1} \left(dx^6 + \sum_{\sigma} F_i^{\sigma} dx^i \right)^2 \right. \\ &\left. + f_7 dx_7^2 + f_8 (dx_8^2 + dx_9^2) + f_6 \tau_2(u) \left| \eta^2(\tau(u)) \prod_{i=1}^{24} \frac{du}{(u - u_i)^{1/12}} \right|^2 \right], \end{aligned} \quad (3.22)$$

which is very suggestive of the multi $D3$ -brane metric provided certain conditions are imposed on (f_1, \dots, f_4) at IR. The condition that we want for our case would be the following obvious one:

$$f_1 V_1^{-1} \approx f_i \quad (3.24)$$

with $i = 2, 3, 4$. This is not too difficult to show. In the far IR, as we discussed above, the system is described by $D3$ -branes probing the geometry instead of the $D5\text{-}\overline{D5}$ pairs. This means that the Taub-NUT geometry is essentially decoupled from the $D3$ -brane geometry implying that the metric seen along the $D3$ -brane directions is given by the first line of (3.16) with $k = 0$ i.e $f_1 = f_i$. Additionally the other warp factor V_1 is defined in terms of the fundamental strings, α_1 , as shown in (3.17). We can always make a Lorentz transformation to go to a frame of reference where only world-volume magnetic fields, $F_{67}^{(1,2)}$, are turned on and the electric field, F_{0i} , is zero (see also footnote 19). Thus $\alpha_1 \rightarrow 0$ and $V_1 \approx 1$, so that (3.24) is satisfied. Therefore our *ansatze* for the IR metric will take the following form:

$$ds_{\text{IR}}^2 = \mathcal{F}_1^{-1/2} ds_{0123}^2 + \mathcal{F}_1^{1/2} ds_{\perp}^2 = \mathcal{F}_1^{-1/2} (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \mathcal{F}_2 |d\vec{w}|^2 \\ + \mathcal{F}_3 \left(dx^6 + \sum_{\sigma} F_i^{\sigma} dx^i \right)^2 + \mathcal{F}_4 \tau_2(u) \left| \eta^2(\tau(u)) \prod_{i=1}^{24} \frac{du}{(u - u_i)^{1/12}} \right|^2, \quad (3.25)$$

where \mathcal{F}_i are related to each other by supergravity EOMs. Thus the non-conformal IR limit does not have any immediate simplification. But if we go to the special arrangements of the seven-branes where we expect constant coupling scenarios [72, 22] then the EOMs connecting \mathcal{F}_i should simplify to give us the near-horizon AdS_5 geometry.

On the other hand, our background (3.20) could also tell us about the UV geometry. At UV we cannot ignore the Taub-NUT singularities and therefore the $D5\text{-}\overline{D5}$ pairs would wrap various vanishing cycles of the multi Taub-NUT geometry. The $D5$ -brane charges cancel, but as we discussed before, the $D3$ -brane charges add up and in fact the $D3$ -branes are *delocalized* along the $x^{6,7}$ directions. From the above

discussions, we now expect the UV metric to be given by:

$$\begin{aligned}
ds_{\text{UV}}^2 &= \frac{1}{\sqrt{V_2}} \left[-f_1 V_1^{-1} dx_0^2 + f_3 dx_1^2 + f_2 dx_2^2 + f_4 dx_3^2 + f_7 V_2 dx_7^2 \right. \\
&\quad \left. + f_5 V_2 V_1^{-1} \left(dx^6 + \sum_{\sigma} F_i^{\sigma} dx^i \right)^2 \right] \\
&\quad + \sqrt{V_2} \left[f_8 (dx_8^2 + dx_9^2) + f_6 \tau_2(u) \left| \eta^2(\tau(u)) \prod_{i=1}^{24} \frac{du}{(u - u_i)^{1/12}} \right|^2 \right]. \quad (3.26)
\end{aligned}$$

Therefore the UV physics is now captured not by a four-dimensional spacetime, but by a six-dimensional spacetime! In the constant coupling scenario of [72, 22] the near-horizon geometry should give us an AdS_7 spacetime. It would be interesting to compare the UV and IR limits with [36] (and the earlier work of [45]).

Before moving further, let us make two comments on the IR metric of (3.25). This will help us to compare our F-theory constructions with the brane constructions in type IIA [79] and the brane network in type IIB [14].

- The above metric (3.25) *cannot* come from a type IIA brane configuration with NS5, D4 and D6-branes. In fact even in the so-called delocalized limit the form (3.25) cannot be recovered. In particular it is not possible to see how the second term in the second line of (3.25) could appear from D6-branes of type IIA⁹.
- As we discussed in the previous subsection 3.5.1, a T-duality along x^4 or x^5 to get the brane network model of [14] is *not* possible because the metric (3.25) has non-trivial dependence along the u -plane! If we delocalize along

⁹ One might observe that a T-duality along the isometry direction of the Taub-NUT space i.e along the x^6 direction, naively leads to a NS5-brane *delocalized* along the x^6 direction. In [48] this issue has been addressed in great details and the final answer reveals an additional dependence of the NS5 harmonic function along the angular x^6 direction (see also [74]). However similar analysis have not been attempted for the seven-branes, and at this stage it is not *a-priori* clear to us how this T-duality should be taken to allow for localized gravitational solutions.

these directions then we can recover the brane network of [14] but will lose all non-trivial information on the u -plane. Therefore the F-theory picture captures more information than the brane network of [14].

Thus from the above comments we see that the F-theory models are in some sense better equipped to capture non-trivial informations of the corresponding gauge theories as the probe branes have direct one-to-one connections to the corresponding gauge theories. The only restriction that we could see in our models has to do with the upper-bound on the *number* of seven-branes. F-theory tells us that the number of seven-branes have to be at most 24 otherwise the singularities on the u -plane will be too drastic to have a good global description [76]. This restriction on the number of seven-branes (or to the global symmetries of the corresponding gauge theories) should not be too much of an issue because one may resort to only local F-theory description assuming that the global completions may be done by introducing anti-branes that would preserve $\mathcal{N} = 2$ supersymmetry up to certain energy scales (see also [53]). The energy scale may be chosen in such a way that all the above discussions may succinctly fit in. The global symmetries in these theories may then be made arbitrarily large so as to encompass most of the Gaiotto's models. It would of course be an instructive exercise to explicitly demonstrate a concrete example with a large global symmetry that, in the Seiberg-Witten sense, remains *integrable*. Once there, the far UV picture of this model should be interesting to unravel from our set-up.

One final thing before we end this subsection is to analyze the background fluxes. At the far IR the six-form charges should cancel completely but at UV they should appear as dipole charges¹⁰. The four-form charges should be quantized and should be proportional to the number of D5- $\overline{\text{D5}}$ pairs. In addition to that there would be a

¹⁰ For D5- $\overline{\text{D5}}$ pairs, the tachyonic behavior emerges at distances of order $\sqrt{\alpha'}$ or less [21]. (See appendix E in [27].) In a tachyon-free system, we expect D5- $\overline{\text{D5}}$ to be separated by a distance larger than that, creating a dipole moment in the system.

background axion-dilaton $\tau(u)$ that is a function on the u -plane, and NS and RR two-forms field with the required three-form field strengths. For the conformal cases we expect $\tau(u)$ to take one of the values given in [72, 22]. These fluxes and branes deform both the Taub-NUT and the seven-brane geometries and together they preserve the required supersymmetry for our case.

3.5.3 Mapping to the conformal cases

The D5- $\overline{\text{D5}}$ -brane pairs at the Taub-NUT singularities also tell us what the UV gauge symmetry should be for our case. Imagine we have a m multi-centered Taub-NUT geometry, then the N D5- $\overline{\text{D5}}$ brane pairs wrapped around the m vanishing cycles lead to mN fractional $D3$ -branes where each of the N fractional $D3$ -branes carry a total RR charge of N/m in appropriate units. Since there are m copies of this, there is a total charge of N $D3$ -branes, leading us to speculate the UV gauge symmetry to be m copies of $SU(N)$, i.e:

$$SU(N) \times SU(N) \times SU(N) \times \cdots \times SU(N). \quad (3.27)$$

Once the wrapped D5- $\overline{\text{D5}}$ pairs are decomposed in terms of fractional $D3$ -branes¹¹, these fractional branes can now freely move along the F-theory u -plane, i.e the Coulomb branch of the theory¹². This is illustrated in figure 3.8. However even the individual set of N fractional branes may separate by further Higgsing to $U(1)^N$. In that case the individual fractional $D3$ -brane carry a net RR charge of $1/m$ in appropriate units.

It is now interesting to see how supersymmetry and global symmetries would constrain the underlying picture. Since the D5-brane charges cancel, the model only has fractional $D3$ -branes and therefore the fractional- $D3$ and seven-branes preserve

¹¹ Recall that there are no D5-brane charges in the background.

¹² Recall that if these $D3$ -branes move along the Taub-NUT directions (i.e the Higgs branch) they become fractional instantons.

the required supersymmetry as we discussed before. However the global symmetries are crucial. So we should look for various arrangements of the seven-branes that allow $\tau(u) = \text{constant}$ in the u -plane. These arrangements should be related to the models studied by Gaiotto [35]. One interesting example is related to the construction that

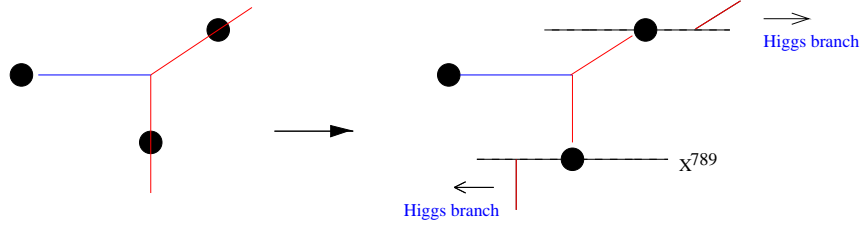


Figure 3.9: The brane junction that we get in the left of the figure may further be truncated by breaking the NS5 and the (p, q) five-brane and moving along the Higgs branch. However as we argue below, this may not be required to see the underlying dualities.

we had in figure 3.6 wherein we showed how the simplest T-dual brane network may come out from our scenario. The D5-brane ends on the seven-brane, and so would any horizontal (i.e along x^6) D5-branes in this scenario. However the NS5-branes and the (p, q) five-branes have to intersect the seven-branes as shown in the left of figure 3.9. From [14] we might expect the figure on the right where parts of the NS5 and the (p, q) five-branes have been moved away along the Higgs branch. This configuration is in principle rather non-trivial to get from the Taub-NUT scenario, but there is no reason for an exact one-to-one correspondence [14] as we argued earlier. The $\mathcal{N} = 2$ dualities should in principle be seen as long as we have the D5-brane configurations right.

To see further how this is implemented let us consider our UV configuration of a three set of three seven-branes wrapping three-centered Taub-NUT manifold with no fractional $D3$ -branes. One immediate advantage of this is that, since there are no fractional $D3$ -branes to start with, a T-dual map to [14] will be easier. The Weierstrass equation governing the background at a given point is given by:

$$y^2 = x^3 + x(c_0 + c_1 z) + b_0 + b_1 z + b_2 z^2 \quad (3.28)$$

where we are choosing the *split* case of the Tate algorithm [73] where the choices of (c_i, b_i) can be read off from eq. (4.7) of [16]. This means that the discriminant locally is of the form

$$\Delta \sim z^3 \quad (3.29)$$

and so if we have three copies of this on the u -plane, we are guaranteed that we will have *no* gauge symmetry but only a global symmetry of

$$SU(3) \times SU(3) \times SU(3). \quad (3.30)$$

In one set of three seven-branes we can first switch on constant A_6 fields so that a T-duality along x^6 may lead to an arrangement of the seven-branes shown in the LHS of figure 3.10. Note however that due to the background axion-dilaton, the T-dual NS5-branes will *not* remain straight. The background axion-dilaton will affect the NS5-branes and they will in turn get bent. This phenomena is exactly what we see for string networks. In [23] it was shown how a network of (p, q) strings get bent in the presence of axion-dilaton.

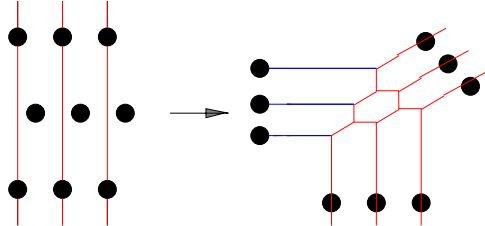


Figure 3.10: On the left is a non-susy configuration that appears from naive T-duality of the Taub-NUT model, as the branch cuts of the seven-brane (shown as solid black circles) would modify the parallel NS5-branes configuration from the local axion-dilaton charges. This is similar to the deformation of a string-network from background axion as shown in [23]. The arrangement of the seven-branes along x^6 direction come from the original framework of the wrapped seven-branes with A_6 switched on. T-duality convert A_6 to x^6 shown on the left. On the right is the susy configuration by moving the seven-branes across the “bent” NS5-branes.

Once this is taken care of, we can switch on a time-varying gauge field on the same set of seven-branes in exactly the similar way we discussed earlier. This would create

fractional $D3$ -branes to cancel the gauge anomalies which, in the T-dual framework, is given by the RHS of figure 3.10. The other two sets of three seven-branes¹³ can be arranged to intersect the NS5 and the (p, q) five-branes. This is also exactly the configuration studied in [14] (with mild differences).

Note that the above configuration is in principle *different* from the configuration of three fractional $D3$ -branes probing seven-branes background where the seven-branes wrap multi-centered Taub-NUT geometry. The T-dual of the m -center Taub-NUT space would be m parallel NS5-branes as above. The UV gauge group will be determined as (3.27) but we may only consider the low energy limit where the Taub-NUT singularities are not prominent. This however doesn't mean that we have recovered the above model because there would still be a remnant gauge symmetry in the model even at far IR. We may play the same game of removing fractional $D3$ -branes by switching on time-varying gauge field on each of the Taub-NUT cycles but the model will not be similar to our earlier case and the gauge theory dynamics will be different.

Coming back to our model, we can now rearrange the seven-branes using F-theory Weierstrass equation to go to another limit with a different global symmetry. This time the Weierstrass equation can be changed from (3.28) to the following local form:

$$y^2 = x^3 + z^4, \quad (3.31)$$

implying a global E_6 symmetry on the gauge theory side. From F-theory side the discriminant locus and the underlying four-fold \mathcal{M}_8 will become respectively:

$$\Delta \sim z^8, \quad \mathcal{M}_8 = \mathbb{R}^4/\mathbf{Z}_3 \times \mathbf{TN}_3. \quad (3.32)$$

Observe that globally the K3 manifold has degenerated to its \mathbf{Z}_3 orbifold limit with a full global symmetry of E_6^3 [22]. For this global symmetry we are indeed at the constant coupling point [22] (see also [60, 61]).

¹³ Clearly not all the seven-branes are $D7$ -branes, as we would need $[p, q]$ seven-branes for consistency with Gauss' law.

In our Taub-NUT picture we have now redistributed the seven-branes now as three sets with *eight* seven-branes in each set. We can move the two set of sixteen seven-branes in the u -plane so that we only allow a global symmetry of E_6 . Our picture can also be supported by the T-dual brane network of [14]. This then would realize the Argyres-Seiberg duality [7].

Yet another example to consider would be to view the $SU(3)$ global symmetry to come from an $SU(4)$ symmetry by Higgsing the **4**. This means we are bringing in another set of seven-branes so that the overall configurations wrap a Taub-NUT with four singularities. The local Weierstrass equation now will be:

$$y^2 = x^3 + x(c_0 + c_1 z) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 \quad (3.33)$$

with special relations between (c_i, b_i) such that we are at the *split* \mathbf{A}_3 case [73]. These relations are worked out in [16] which the readers may look up for more details. The discriminant locus is as expected:

$$\Delta \sim z^4, \quad (3.34)$$

so that we have a global $SU(4)$ symmetry. As before if we make three copies of this we will have the required global symmetry of $SU(4)^3$. This configuration maps directly to the brane network studied in [14] so we don't have to go through the details. It suffices to point out that the rearranged seven-branes may now give a global symmetry of (see also [22])

$$E_7 \times E_7 \times SO(8), \quad (3.35)$$

so that the axion-dilaton remains constant throughout the u -plane. Locally near one of the E_7 singularity the F-theory manifold is typically an orbifold of the form:

$$\mathbb{R}^4/\mathbb{Z}_4 \times \mathbf{TN}_4, \quad (3.36)$$

which means that our K3 has become a \mathbb{Z}_4 orbifold of the four-torus.

The above decomposition of the underlying K3 manifold into its various orbifold limits give us a hint what the next configuration would be. This would be the \mathbb{Z}_6 orbifold of the four-torus so that the conformal global symmetry should be [22]

$$E_8 \times E_6 \times SO(8). \quad (3.37)$$

Now since the \mathbb{Z}_6 orbifold creates a deficit angle¹⁴ of at most $\frac{5\pi}{3}$ we know that this is an orbifold with a fixed point of order 6. Therefore our starting point would be to put three copies of six seven-branes wrapping a Taub-NUT with six-singularities leading to an $SU(6)^3$ global symmetry. This then clearly enhances to (3.37) with the local F-theory four-fold given by:

$$\mathbb{R}^4/\mathbb{Z}_6 \times \mathbf{TN}_6. \quad (3.38)$$

The above set of configurations were studied without incorporating any D5- $\overline{\text{D5}}$ -branes in the background. Once we introduce the probes we will not only have global symmetry, but also gauge symmetry. A special rearrangement of the seven-branes may help us to study the conformal theories leading to other Gaiotto dualities. We will discuss a more detailed mappings to these cases in the sequel.

3.5.4 Beyond the conformal cases

Since our model is a direct construction in F-theory, all informations of the type IIB background under non-perturbative corrections are transferred directly to the D3-brane probes. This in particular means that arrangements of the seven-branes that lead to non-trivial axion-dilaton backgrounds would also be transferred to the D3-brane probes, except now they would appear as non-conformal theories on the D3-branes. A simple non-conformal deformation, with our set-up discussed in the previous subsection, is given in figure 3.11. This could be generated from $SO(8)$ in

¹⁴ Recall that for a singularity to be of an orbifold type the deficit angle has to be $2\pi \left(1 - \frac{1}{n}\right)$ for a fixed point of order n .

(3.35) breaking completely to $SU(2)$ by first going to $SO(7)$ and then $SO(7)$ breaking to $SU(2) \times SU(2) \times SU(2)$. Recall from [34] that¹⁵

$$SO(8) \equiv \mathbf{A}^4 \mathbf{B} \mathbf{C}, \quad (3.39)$$

so that the perturbative pieces generate the subgroup $SU(2) \times SU(2)$. Separating the $[0, 1]$ and the $[1, -1]$ seven-branes from the bunch of the six seven-branes allow us to achieve this. Once we further break the other $SU(2)$, we can easily generate the **248** of E_8 from (3.35) via:

$$\mathbf{248} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{133}) \oplus (\mathbf{2}, \mathbf{56}). \quad (3.40)$$

This is clearly a non-conformal deformation in the Taub-NUT background as the axion-dilaton-dilaton values are no longer constant in the u -plane. It is interesting to note that if we take other model (3.37) then there exist a limit where the non-conformal deformation in this model is precisely the non-conformal deformation of the earlier case. This is when $SO(8)$ in (3.37) is completely broken to $U(1)$ by moving all the \mathbf{A} , \mathbf{B} and \mathbf{C} branes except one \mathbf{A} brane. Under this circumstances the **56** of E_7 is easily generated from (3.37) for the $D3$ -brane probes to see identical physics as the earlier case:

$$\mathbf{56} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{27} \oplus \overline{\mathbf{27}}. \quad (3.41)$$

¹⁵ In the following \mathbf{A} , \mathbf{B} and \mathbf{C} are the three monodromy matrices given as:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} 4 & 9 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{C} \equiv \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

These monodromy matrices are derived from the monodromies around $D7$ and the two (p, q) seven-branes in figure 4.8 respectively. For more details the authors may refer to [34].

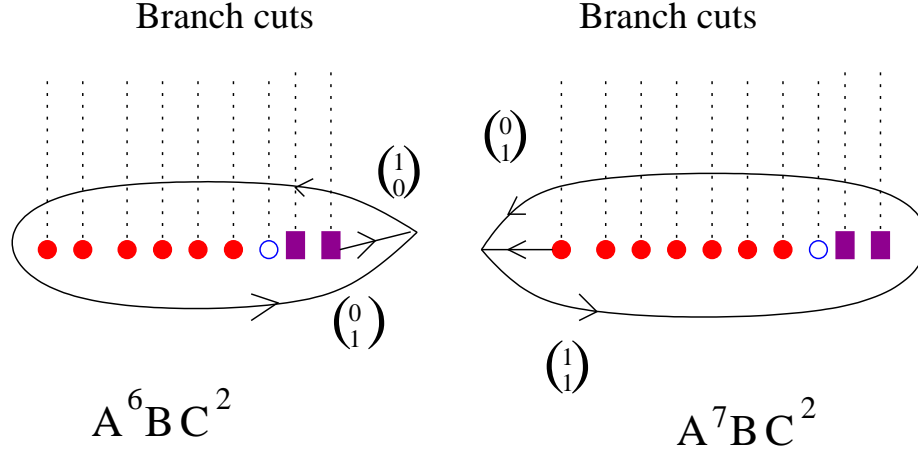


Figure 3.11: A simple non-conformal deformation of the theory where the multiple D5- $\overline{\text{D5}}$ branes are probing a $E_7 \times E_8$ singularity. The global symmetry will have additional $U(1)$ that are not shown in the figure. In the language of [34] the filled circles are A -branes, the empty circles are B -branes and the filled squares are C -branes.

Finally, to see similar non-conformal deformation from the first E_6^3 model that we studied above, we can go back to the *unenanced* case for one of the E_6 group, namely the $SU(3)^3$ global symmetry with the particular arrangements of the seven-branes as in figure 3.10. For this case the **56** of E_7 is generated as (3.41), but the **248** of E_8 is now generated via:

$$\mathbf{248} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}), \quad (3.42)$$

provided of course that the remnant subgroup of $SU(3) \times SU(2) \times U(1)$ is completely broken. Only under this case the physics seen by the $D3$ -brane probes will be identical.

The above non-conformal deformations were extensions of the conformal theories with exceptional global symmetries. They aren't the simplest non-conformal models that we could study here. There exist simpler models if we introduce, in addition to the D5- $\overline{\text{D5}}$ -brane probes, some *additional* D5-branes wrapping vanishing 2-cycles of the Taub-NUT space.

Let us take a concrete example where we have k D5- $\overline{\text{D5}}$ -brane pairs at a point in the Taub-NUT space with m singularities. In addition to these probes, let us also introduce M_i (with $i = 1, \dots, m$) D5-branes wrapping the m vanishing 2-cycles of the

Taub-NUT space. It is immediately clear that the gauge symmetry now will change from (3.27) to the following:

$$\prod_{i=1}^m SU(k + M_i) \equiv SU(k + M_1) \times SU(k + M_2) \times \cdots \times SU(k + M_m). \quad (3.43)$$

The above theory is obviously non-conformal as the additional wrapped D5-branes break the conformal invariance already in the absence of any flavor symmetry. If we take the Taub-NUT and wrap M D5-branes on the vanishing 2-cycle, then the gauge group will be special case of (3.43), namely

$$SU(k + M) \times SU(k). \quad (3.44)$$

This brings us exactly to the cascading models of [67, 45, 2, 4, 15] where the authors have argued cascading behavior in this model (see [4, 15] for a more recent study)! It is then clear that our model can have an even more interesting cascading dynamics because there is an option of having a much bigger gauge group as can be seen from (3.43). Furthermore due to the presence of seven-branes, the cascading model is of the Ouyang type [63, 19]. This means there is a chance that cascades would be *slowed* down by the presence of fundamental flavors, much like the one studied in [63, 19].

The connection to $\mathcal{N} = 1$ cascade [54] is now clear: we can break the $\mathcal{N} = 2$ supersymmetry by non-trivially fibering the Taub-NUT over the compactified u -plane. Writing $u \equiv z_4 = x^4 + ix^5$, the fibration is explicitly:

$$z_1^2 + z_2^2 + z_3^2 = -z_4^2 \equiv -u^2, \quad (3.45)$$

which is an ALE space with coordinates (z_1, z_2, z_3) fibered over the u -plane. Near the node $x^4 = x^5 = 0$ the geometry is our familiar Taub-NUT space, and the equation (3.45) is a conifold geometry. Once this is achieved the $\mathcal{N} = 1$ cascading behavior can take over with the termination point governed by *either* a confining theory, or a conformal theory depending on the choice of the flavor symmetry.

This concludes our discussion about the connection between a class of Gaiotto models and F-theory with a multi Taub-NUT space. We seem to have provided a

geometry with all the right ingredients to generate a cascade for $\mathcal{N} = 2$ non-conformal supersymmetric gauge theories in four dimensions. Limited by the complexity of type IIB/F-theory language, we can not, at this point, push further our analysis of the dynamic of this cascade. For this reason, we will turn, in the next chapter, to the type IIA/M-theory language where the origin of $\mathcal{N} = 2$ cascade will becomes clarified.

Chapter 4

Type IIA perspective on cascading gauge theories

4.1 Introduction

The system of p $D3$ -branes and M $D5$ -branes at the tip of the conifold in type IIB string theory [54] exhibits many non-trivial phenomena such as confinement, dynamical symmetry breaking and a rich landscape of ground states. It is also an important example of gauge/gravity duality, which plays a role in studies of string phenomenology and early universe cosmology.

The low energy effective field theory of this system is an $\mathcal{N} = 1$ supersymmetric four dimensional gauge theory with gauge group $SU(M + p) \times SU(p)$ and matter in the bifundamental representation [54]. The rich vacuum structure of this gauge theory was described in [30]. It can be interpreted in terms of a “duality cascade” – a sequence of gauge theories with varying ranks which provide a description of the different vacua. Some of these vacua have a regular type IIB supergravity description, which has also been extensively investigated.

The T-dual of the type IIB construction of [54] is given by a system of $NS5$ -branes and $D4$ -branes in type IIA string theory. This system was mentioned in [54] and further studied in [3]. One of our goals below will be to build on the results of [3]

and reproduce the results of [54, 30] using the IIA description. We will also discuss some non-supersymmetric aspects of the dynamics.

We will see that the IIA description provides a nice picture of the supersymmetric and non-supersymmetric vacua. As is standard in studying brane dynamics in string theory, the three descriptions (gauge theory, IIA and IIB) are valid in different regions in the parameter space of the brane system. This should not matter for the supersymmetric vacuum structure, and indeed we will reproduce the results of [54, 30] in the IIA language. Many aspects of the non-supersymmetric vacuum structure are also expected to agree, and we will find that to be the case.

Cascading behavior was found in a wide variety of theories, some of which do not exhibit Seiberg duality. We will briefly discuss an example of this phenomenon, an $\mathcal{N} = 2$ supersymmetric quiver theory closely related to $\mathcal{N} = 2$ SQCD, and use the IIA description to identify the origin of the cascade in this theory.

In gauge theory there are actually two versions of Seiberg duality. The strong version asserts that the electric and magnetic theories of [71] are equivalent in the infrared at the origin of moduli space and in the absence of deformations of the Lagrangian. In general one or both of these (conformal) theories are strongly coupled, and their equivalence has not been proven to date. The weaker version concerns the infrared equivalence of the two theories in the presence of deformations, and/or along moduli spaces of flat directions. In this case, one can often analyze the long distance behavior of both theories precisely and show their equivalence. Examples of this were studied in the original work of [71] and many subsequent papers. A discussion in a context closely related to the cascading gauge theory appears in [43]. In the IIA brane description of Seiberg duality [31], the strong version of Seiberg duality involves exchanging NS and NS' -branes connected by $D4$ -branes, which involves fivebranes intersecting at a point in the extra dimensions. The statement of duality is that this process is smooth, which is non-trivial and unproven to date. The weak version of the duality involves smooth deformations of the brane system, which obviously do not change the low energy behavior. The discussion below will make it clear that the cascading gauge theory only requires the weak version of the duality. This is why it

is manifest in the brane description. It also will clarify that while the authors of [30] used Seiberg duality to derive the vacuum structure of the model, one should be able to do this without that assumption, and show that the resulting vacuum structure exhibits the correct duality structure.

The plan of the paper is as follows. In section 4.2 we introduce the classical $\mathcal{N} = 1$ supersymmetric gauge theory and IIA brane system that reduces to it at low energies. We review the structure of the classical moduli spaces in both languages, and show that they agree. We also discuss some non-supersymmetric vacua that appear for non-zero Fayet-Iliopoulos (FI) coupling.

In section 4.3 we discuss the quantum theory. We show that the quantum moduli space of the brane system is the same as that of the gauge theory, and in particular exhibits the cascading behavior found in [54, 30]. The brane picture gives a simple description of the cascade and helps understand which vacua of theories with different values of p agree, and which do not. In this picture, the cascade is associated with the fact that for a given value of the UV cutoff the fivebrane in general winds around a circle. As one reduces the UV cutoff, the winding number decreases. This corresponds in the field theory language to decreasing p by a multiple of M . Vacua in which the fivebrane does not wind (or does not wind enough times) around the circle are not in general the same in theories with different values of p .

In section 4.4 we discuss non-supersymmetric vacua of the brane system. We show that for the non-supersymmetric vacua that appear for non-zero FI coupling the quantum brane picture incorporates chiral symmetry breaking, which is expected to occur in the corresponding low energy gauge theory. We also discuss the IIA analog of the metastable states discussed in the IIB language in [52]. We find that these states are present in the brane system, but the barrier that separates them from the supersymmetric states goes to infinity in the field theory limit. Thus, they become stable in that limit.

In section 4.5 we discuss the $\mathcal{N} = 2$ supersymmetric analog of the cascading gauge theory, and in particular address the question how a theory that does not have Seiberg duality can have a duality cascade. We show that the situation is similar to that in

the $\mathcal{N} = 1$ case – the gauge theory has a rich set of vacua, some of which exhibit cascading behavior. Even in these vacua, different theories along the cascade differ by abelian factors in the gauge group.

Section 6 contains a brief discussion of our results; an appendix summarizes some aspects of the IIA description of quantum $\mathcal{N} = 2$ SQCD, which are useful for the discussion in section 4.5.

4.2 Classical theory

We start with a brief description of the classical gauge theory and the corresponding type IIA brane system. We refer the reader to [30, 40] for a more detailed discussion of the two topics. We will draw heavily on the results described in these papers.

As mentioned in the introduction, we will be studying an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group

$$G = SU(N_1) \times SU(N_2) \quad (4.1)$$

with

$$N_1 = M + p; \quad N_2 = p \quad (4.2)$$

The matter consists of chiral superfields $A_{\alpha i}^a$ and $B_{\dot{\alpha} a}^i$, where $i = 1, \dots, N_1$, $a = 1, \dots, N_2$, are gauge indices, and $\alpha, \dot{\alpha} = 1, 2$ are global symmetry labels. As implied by the notation, the matter fields transform under G as follows:

$$A_{\alpha} \quad (N_1, \bar{N}_2) \quad (4.3)$$

$$B_{\dot{\alpha}} \quad (\bar{N}_1, N_2) \quad (4.4)$$

There is also a tree level superpotential,

$$W_0 = \frac{h}{2} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} A_{\alpha i}^a B_{\dot{\alpha} b}^i A_{\beta j}^b B_{\dot{\beta} a}^j = h (A_{1i}^a B_{1b}^i A_{2j}^b B_{2a}^j - A_{1i}^a B_{2b}^i A_{2j}^b B_{1a}^j). \quad (4.5)$$

To compare to standard discussions of $\mathcal{N} = 1$ SQCD, it is useful to note that the $SU(N_1)$ factor in the gauge group (4.1) “sees” $2N_2$ flavors, and similarly for $SU(N_2)$. Since $N_1 > N_2$ (4.2), in the quantum theory the quartic superpotential (4.5) is a relevant perturbation of the IR fixed point of the $SU(N_1)$ gauge theory obtained by turning off the $SU(N_2)$ gauge coupling, and an irrelevant perturbation of the corresponding $SU(N_2)$ fixed point.

The global symmetry of the model includes $SU(2) \times SU(2)$, with the two factors acting on the indices α and $\dot{\alpha}$, and a $U(1)$ symmetry which assigns charges $+1$ to A and -1 to B ; this symmetry is usually referred to as baryon number. We will denote it by $U(1)_b$ and will mostly consider the theory in which it is gauged, since this is the case in the IIA brane system we will study. It is also useful to consider this case in the IIB theory, since it is relevant to the embedding of the conifold geometry in a compact Calabi-Yau manifold.

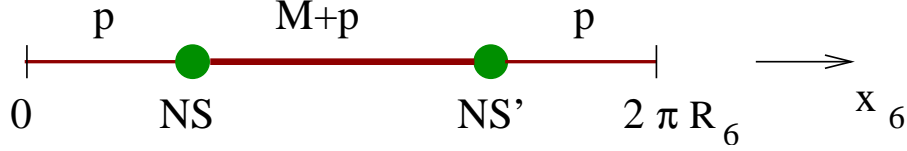


Figure 4.1: The IIA brane configuration that realizes the cascading gauge theory. The $NS5$ -branes are depicted in green, and are connected by $M+p$ $D4$ -branes on one side of the x^6 circle (whose radius is R_6) and by p $D4$ -branes on the other

The IIA brane system that gives rise to the above gauge theory at low energies is depicted in figure 4.1. The system contains two kinds of branes: $NS5$ -branes localized on a circle $x^6 \sim x^6 + 2\pi R_6$, represented by green circles in the figure, and stacks of $D4$ -branes connecting them, represented by red lines. The orientations of the different branes in the $9+1$ dimensional spacetime are as follows:

$$NS : \quad 012345 \quad (4.6)$$

$$NS' : \quad 012389 \quad (4.7)$$

$$D4 : \quad 01236 \quad (4.8)$$

This configuration preserves $\mathcal{N} = 1$ SUSY in the $3 + 1$ dimensions common to all the branes, (0123). The low energy effective theory of this brane system contains $U(M + p) \times U(p)$ $\mathcal{N} = 1$ SYM, associated with the massless excitations living on the $M + p$ and p $D4$ -branes, respectively, and chiral superfields A, B (4.3), which come from strings connecting the two stacks of $D4$ -branes. An overall $U(1)$ in $U(M + p) \times U(p)$ is decoupled, and can be ignored, but the relative $U(1)$ is precisely the $U(1)_b$ discussed above. Hence, the brane construction gives the theory in which this symmetry is gauged, as mentioned above.

The classical $U(N_i)$ gauge couplings are determined by the lengths of the corresponding branes, L_i ,

$$\frac{1}{g_i^2} = \frac{L_i}{g_s l_s} \quad (4.9)$$

As is clear from figure 4.1,

$$L_1 + L_2 = 2\pi R_6, \quad i.e. \quad \frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{2\pi R_6}{g_s l_s} \quad (4.10)$$

Our purpose in the remainder of this section is to compare the classical moduli space of supersymmetric vacua of the gauge theory, studied in [30], to that of the brane system. We will also discuss some non-supersymmetric vacua of the theory. In the next section we will describe the quantum moduli space.

The D-term equations of this gauge theory can be written as the following matrix equations for the $(p + M) \times p$ matrices $A_\alpha, B_{\dot{\alpha}}^\dagger$:

$$\sum_{\alpha} A_{\alpha} A_{\alpha}^{\dagger} - \sum_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} B_{\dot{\alpha}} = \frac{\mathcal{U}}{p} I_p \quad (4.11)$$

$$\sum_{\alpha} A_{\alpha}^{\dagger} A_{\alpha} - \sum_{\dot{\alpha}} B_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} = \frac{\mathcal{U}}{M + p} I_{M+p} \quad (4.12)$$

with I_n an $n \times n$ identity matrix, and

$$\mathcal{U} = \text{Tr} \left(\sum_{\alpha} A_{\alpha} A_{\alpha}^{\dagger} - \sum_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} B_{\dot{\alpha}} \right) \quad (4.13)$$

In the theory with gauged $U(1)_b$, one must set $\mathcal{U} = 0$; turning on a Fayet-Iliopoulos (FI) term ξ for this $U(1)$, modifies this to

$$\mathcal{U} = \xi \quad (4.14)$$

Classical supersymmetric vacua correspond to solutions of the D-term equations (4.11) – (4.14) as well as the F-term conditions for the superpotential (4.5). For general M , p , and setting $\xi = 0$ for now, the solutions of these equations can be written, up to gauge transformations, in the diagonal form

$$A_\alpha = \begin{pmatrix} A_{\alpha 1}^1 & & & \\ & A_{\alpha 2}^2 & & \\ & & A_{\alpha 3}^3 & \\ & & & \ddots \\ & & & & A_{\alpha p}^p \end{pmatrix}; B_{\dot{\alpha}}^t = \begin{pmatrix} B_{\dot{\alpha} 1}^1 & & & \\ & B_{\dot{\alpha} 2}^2 & & \\ & & B_{\dot{\alpha} 3}^3 & \\ & & & \ddots \\ & & & & B_{\dot{\alpha} p}^p \end{pmatrix} \quad (4.15)$$

The eigenvalues $A_{\alpha a}^a$ and $B_{\dot{\alpha} a}^a$, $a = 1, \dots, p$ satisfy the constraints

$$\sum_{\alpha} |A_{\alpha a}^a|^2 - \sum_{\dot{\alpha}} |B_{\dot{\alpha} a}^a|^2 = 0 \quad (4.16)$$

For given a , the eigenvalues are four complex fields, which satisfy one real constraint (4.16). Another real field (for each a) is removed by the (Higgsed) gauge symmetry. Thus, the moduli space is $3p$ (complex) dimensional. It can be described by the $4p$ complex coordinates

$$z_{\alpha \dot{\alpha}}^a = A_{\alpha a}^a B_{\dot{\alpha} a}^a \quad (4.17)$$

which satisfy the (complex) constraints

$$\det_{\alpha \dot{\alpha}} z_{\alpha \dot{\alpha}}^a = 0 \quad (4.18)$$

Together with the symmetry of permutation of the p eigenvalues, we conclude that the classical moduli space is a symmetric product of p copies of the singular conifold

(4.18),

$$\mathcal{M}_0 = \text{Sym}_p(\mathcal{C}_0) \quad (4.19)$$

At a generic point in the moduli space, the low energy theory consists of an $SU(M)$ $\mathcal{N} = 1$ SYM theory¹

In terms of the brane system of figure 4.1, the moduli space described above is obtained by noting that the configuration contains p $D4$ -branes that wrap the circle, and can thus freely move in the \mathbb{R}^5 labeled by (45789); another (compact) dimension of moduli space is obtained from a component of the gauge field on the fourbranes, A^6 . A generic point in the moduli space is described in figure 4.2. The p mobile branes support $U(1)^p$ $\mathcal{N} = 4$ SYM, while the M localized branes give rise to pure $\mathcal{N} = 1$ SYM with gauge group $SU(M)$ (and a decoupled $U(1)$ mentioned above), in agreement with the gauge theory analysis.

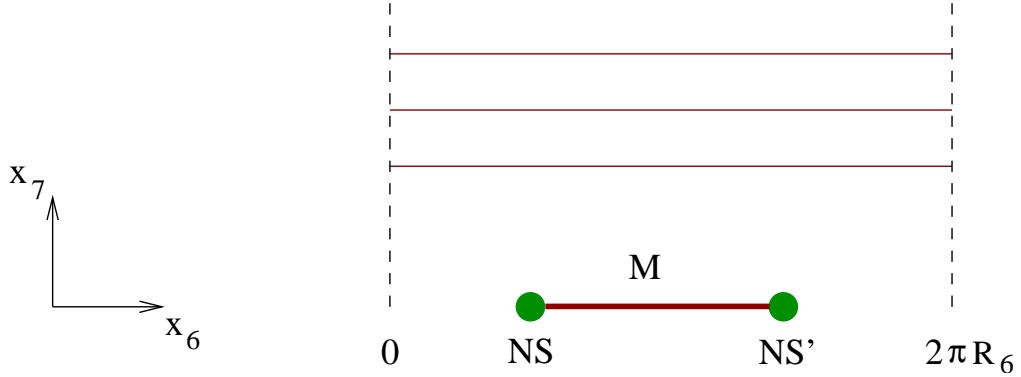


Figure 4.2: A generic point in the classical moduli space. p $D4$ -branes wrap the x^6 circle and can move in the transverse space; M are stretched between the fivebranes and give rise to $SU(M)$ $\mathcal{N} = 1$ SYM

The form of the moduli space (4.19) is very natural from the brane perspective: in the classical gauge theory limit, the separation between the fivebranes goes to

¹ The unbroken $SU(M)$ is a subgroup of the $SU(M+p)$ factor in (4.1) and p copies of $\mathcal{N} = 4$ SYM with gauge group $U(1)$.

zero [40], and the M $D4$ -branes in figure 4.2 4.2 can be ignored. The fivebranes are described by the equation $vw = 0$, where

$$v = x^4 + ix^5 ; \quad w = x^8 + ix^9 \quad (4.20)$$

This is known to be a dual description of the conifold (obtained by T-duality in x^6 ; see e.g [75, 24, 59]). Under this T-duality, the mobile $D4$ -branes turn into $D3$ -branes living on the conifold, in agreement with (4.19).

We next turn to the case where the FI parameter ξ (4.14) is non-vanishing. In general, supersymmetry is then broken, with vacuum energy $V \sim g^2 \xi^2$ [30]. The case

$$p = kM \quad (4.21)$$

with integer k is special. In that case the gauge theory has an isolated supersymmetric vacuum, in which for $\xi > 0$ one has $B_{\dot{\alpha}} = 0$,

$$A_{\alpha=1} = C \begin{pmatrix} \sqrt{k} & 0 & 0 & . & 0 & 0 \\ 0 & \sqrt{k-1} & 0 & . & 0 & 0 \\ 0 & 0 & \sqrt{k-2} & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & . \end{pmatrix} \quad (4.22)$$

and

$$A_{\alpha=2} = C \begin{pmatrix} 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & \sqrt{2} & . & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & \sqrt{k} \end{pmatrix} \quad (4.23)$$

Each entry in the matrices (4.22), (4.23) is proportional to an $M \times M$ unit matrix, and the constant C satisfies (4.13), (4.14), $\xi = k(k+1)M|C|^2$. For $\xi < 0$, one finds a similar vacuum with $A \leftrightarrow B$. The low energy theory in the baryonic vacuum is

described by an unbroken $SU(M)$ $\mathcal{N} = 1$ SYM, but unlike the mesonic branch, this $SU(M)$ is embedded non-trivially in both factors of the gauge group G (4.1).

In the theory with gauge group $SU((k+1)M) \times SU(kM)$ (i.e with ungauged baryon number), (4.22), (4.23) give rise to a one complex dimensional moduli space of vacua labeled by C , which is usually referred to as the baryonic branch. Our interest is in the theory where $U(1)_b$ is gauged, in which (classically) it only appears for non-zero ξ and is isolated.

In the brane system, the FI coupling ξ corresponds to the relative displacement between the fivebranes in x^7 [40]. It is clear that generically this breaks supersymmetry, leading to configurations such as that of figure 4.3 , in which different $D4$ -branes are not mutually BPS.

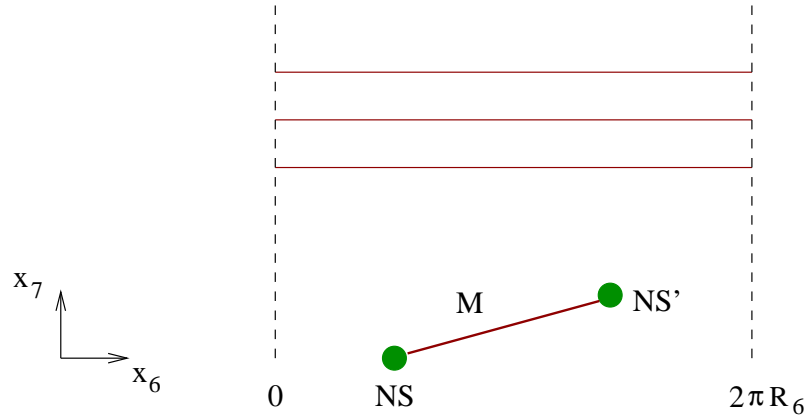


Figure 4.3: Turning on an FI term in general leads to branes at an angle and breaks SUSY.

The baryonic vacuum (4.22), (4.23) is described in terms of the branes by the configuration of figure 4.4. The red line corresponds to a stack of M $D4$ -branes, which connects the NS and NS' -branes, in the process winding k times around the circle. It is easy to check that all the branes in figure 4.4 are mutually BPS and the configuration is supersymmetric. Note that the vacuum of figure 4.4 is isolated, as expected from the gauge theory analysis. Turning off the FI term, i.e taking the NS' -brane in figure 4.4 to the x^6 axis, leads to the configuration of figure 4.1 , with

$p = kM$. Thus, the baryonic vacuum coincides in this case with the origin of the mesonic branch, as in gauge theory.

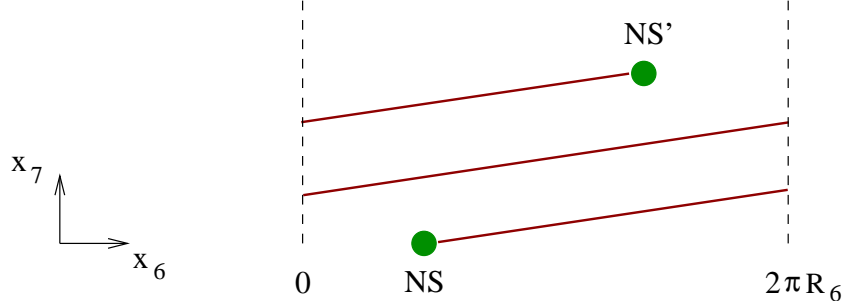


Figure 4.4: The baryonic vacuum for $\xi \neq 0$ corresponds to a stack of M $D4$ -branes connecting the fivebranes and winding k times around the circle ($k = 2$ in the figure).

It is also clear from the figure that the low energy theory in this vacuum is pure $\mathcal{N} = 1$ SYM with gauge group $SU(M)$, and that this $SU(M)$ is non-trivially embedded in the full gauge symmetry $SU((k+1)M) \times SU(kM)$. A quick way to find this embedding is to calculate the gauge coupling of the unbroken $SU(M)$, g . Taking $\xi \rightarrow 0$, the coupling is related to the length of the branes, as in (4.9):

$$\frac{1}{g^2} = \frac{L_1 + 2\pi k R_6}{g_s l_s} = \frac{k+1}{g_1^2} + \frac{k}{g_2^2} \quad (4.24)$$

Thus, the unbroken gauge group is the diagonal $SU(M)$ subgroup of an $SU(M)^{k+1}$ subgroup in $SU((k+1)M)$ and an $SU(M)^k$ subgroup of $SU(kM)$, in agreement with the gauge theory.

Finally, figure 4.4 makes it clear that by moving the NS' -brane in x^6 , we can change the winding number of the spiraling $D4$ -branes, and thus k , by one or more units, without changing the low energy theory² This is a classical precursor of Seiberg

² Of course, if we keep the parameters L , R_6 fixed in the process, the (classical) gauge coupling of the $SU(M)$ gauge theory changes, but we can adjust these parameters so that it does not.

duality [71], which is known to play an important role in the quantum dynamics of the cascading gauge theory. The way it appears here is reminiscent of the discussion of [31]. We will discuss its quantum analog in the next section.

In addition to the supersymmetric vacuum of figure 4.4 (or eqs (4.22), (4.23) in gauge theory) the brane system has a series of non-supersymmetric vacua labeled by the winding number of the M $D4$ -branes stretched between the fivebranes, $l = 0, 1, 2, \dots, k$. The vacuum with winding number l contains M $D4$ -branes connecting the NS and NS' -branes while winding l times around the circle, and $p - lM$ mobile $D4$ -branes. The vacuum with $l = 0$ is the one described in figure 4.3, while that with $l = k$ corresponds to figure 4.4 (and is supersymmetric for $p = kM$).

Since the vacua with $l < k$ are not supersymmetric, it is natural to ask what is the potential on the $3(p - lM)$ complex dimensional pseudomoduli space. Far along the moduli space (i.e for large A , B in (4.16), or z in (4.17)) and at weak IIA string coupling, it is clear that the leading effect is due to closed string exchange between the mobile fourbranes and those stretching between the fivebranes. Since these branes are not parallel, the gravitational attraction does not precisely cancel the RR repulsion, and there is a net attractive force pulling the mobile branes towards the localized ones.

The resulting dynamics facilitates a change in l , as demonstrated in figure 4.5, in which a stack of M mobile $D4$ -branes is pulled towards the fivebranes (a); when it intersects them (b), the brane configuration has an instability towards reconnection (c), due to the presence of an open string tachyon living at the intersection. Condensation of this tachyon leads to the configuration (d), in which l has increased by one unit. The endpoint of this process is the supersymmetric vacuum, with $l = k$ (figure 4.4)³

³ One might think that there are other non-supersymmetric but locally stable ground states in which the winding numbers of the different fourbranes connecting the fivebranes are different and the $SU(M)$ gauge symmetry is broken, but this is not the case.

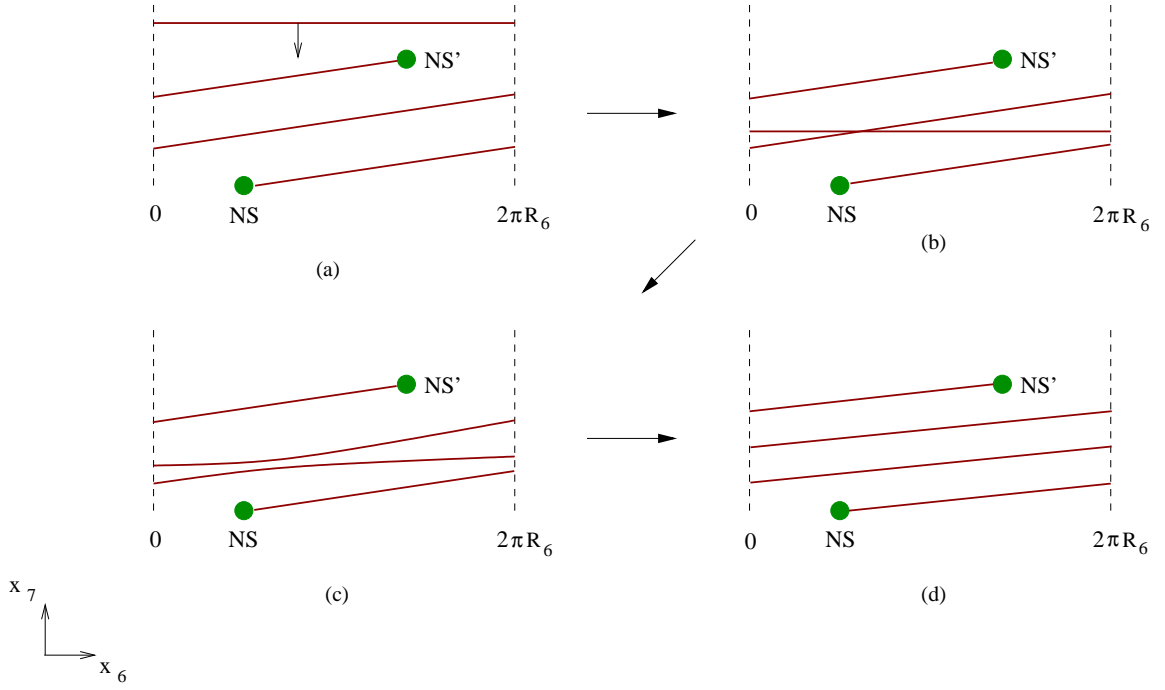


Figure 4.5: Open string tachyon condensation connects vacua with different values of l .

The above brane discussion has a gauge theory counterpart. The F and D term potential at non-zero ξ has a series of non-supersymmetric vacua in which the matrices A and B split into a block of size $(l+1)M \times lM$ in which they look like (4.22), (4.23) with $k \rightarrow l$, and a block of size $p-lM$, which looks like (4.15). The eigenvalues (4.16) label the pseudomoduli space as in the discussion around (4.17). The potential for the pseudomoduli is classically flat, however near the origin of pseudomoduli space there is a tachyonic instability in a different direction in field space, which takes the system towards the supersymmetric vacuum (4.22), (4.23).

A natural question is what is the field theory analog of the classical gravitational attraction that in the brane description gives a potential on pseudomoduli space and leads to the isolated baryonic vacuum of figure 4.4. In other closely related brane systems, such as those that appear in the discussion of the ISS model, this potential is the Coleman-Weinberg (CW) potential computed in [51]. It is natural to expect the same to happen here; we will leave a detailed analysis to future work.

The gravitational brane attraction can be described from the field theory point of view in terms of a non-canonical Kahler potential for the light fields. As discussed in [41, 42, 44], this effect is not identical to the CW potential. The two are dominant in different regions in the parameter space of the brane system, but tend to lead to similar dynamics.

So far we focused on the case $p = kM$ (4.21), but it is easy to generalize to

$$p = kM + \tilde{p}; \quad 1 \leq \tilde{p} \leq M - 1 \quad (4.25)$$

Most of the discussion of this case is the same as before. After all but \tilde{p} of the mobile $D4$ -branes have combined with the localized fourbranes via the process of figure 4.5, we are left with $\tilde{p} < M$ branes in the bulk. These branes are also attracted to the spiraling fourbrane and undergo a process similar to that of figure 4.5, except now it affects only \tilde{p} of the M spiraling fourbranes. This leads to a state in which we have $M - \tilde{p}$ fourbranes which stretch between the fivebranes while winding k times around the circle, and \tilde{p} fourbranes which wind $k + 1$ times (see figure 4.6). Clearly, this state is not supersymmetric (for generic \tilde{p}). We see that in this case, turning on an FI term causes the moduli space to collapse to an isolated non-supersymmetric vacuum.

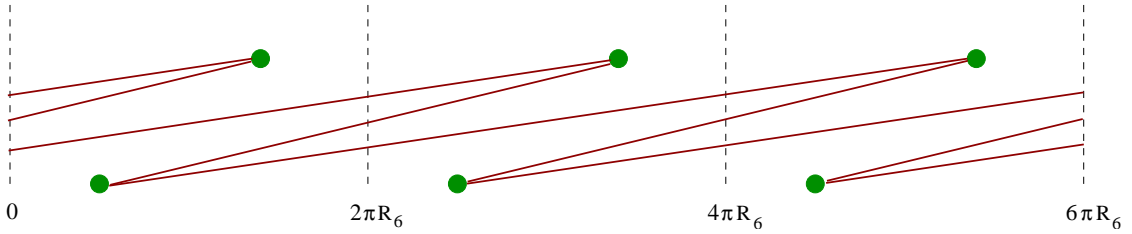


Figure 4.6: The ground state of the brane system with non-zero ξ , viewed in the covering space of the x^6 circle. $M - \tilde{p}$ fourbranes have winding k , while \tilde{p} have winding $k + 1$. The specific case exhibited is $k = \tilde{p} = 1$, $M = 2$, $p = 3$.

The low energy dynamics of the theory with non-zero \tilde{p} can be read off from figure 4.6. The unbroken gauge group is $SU(M - \tilde{p}) \times SU(\tilde{p}) \times U(1)_b$. The embedding of this group in (4.1) can be determined in a similar way to the discussion around (4.24).

The gauge group can be written as

$$SU(M + p) \times SU(p) = SU((k + 1)(M - \tilde{p}) + (k + 2)\tilde{p}) \times SU(k(M - \tilde{p}) + (k + 1)\tilde{p}) \quad (4.26)$$

The first factor contains a $[SU(M - \tilde{p})]^{k+1} \times [SU(\tilde{p})]^{k+2}$ subgroup; the second contains $[SU(M - \tilde{p})]^k \times [SU(\tilde{p})]^{k+1}$. The gauge group corresponding to figure 4.6 involves the diagonal $SU(M - \tilde{p}) \times SU(\tilde{p})$ of all these factors.

There are two kinds of matter fields. One comes from strings both of whose ends lie on the same stack of fourbranes. In addition to the gauge fields, these give fermions in the adjoint representation of the gauge group, which in the absence of the second stack of fourbranes would be the gauginos of an $\mathcal{N} = 1$ supersymmetric model. The second comes from open strings stretched between the two stacks, and is localized at their intersections. As shown in [39], two $D4$ -branes ending on an $NS5$ -brane at a generic angle give rise to a massless Dirac fermion⁴. Thus, the vacuum of the theory of figure 4.6 contains fermions in the bifundamental of $SU(M - \tilde{p}) \times SU(\tilde{p})$ charged under $U(1)_b$, i.e the light matter is similar to that of the original cascading gauge theory, without the scalars.

So far we discussed the non-supersymmetric vacuum of the theory with generic \tilde{p} from the point of view of the IIA brane construction, but it is easy to repeat the discussion in the gauge theory language. For $\xi > 0$, the vacuum field configuration is obtained by splitting each $M \times M$ block on the diagonal in (4.22), (4.23) into blocks of size $M - \tilde{p}$ and \tilde{p} . Looking back at (4.26) we see that in the blocks of size $M - \tilde{p}$ we should use the ansatz (4.22), (4.23), with $C = C_{M-\tilde{p}}$. In the blocks of size \tilde{p} we should use a similar ansatz, with $k \rightarrow k + 1$ and $C = C_{\tilde{p}}$. The D-term potential takes

⁴ The lightest bosonic fields have a mass that depends on the angle between the fourbranes and is non-zero unless these branes are parallel.

the form (up to an overall constant)

$$V_D \simeq k(M - \tilde{p}) \left(|C_{M-\tilde{p}}|^2(k+1) - \frac{\xi}{p} \right)^2 + (k+1)\tilde{p} \left(|C_{\tilde{p}}|^2(k+2) - \frac{\xi}{p} \right)^2 \quad (4.27)$$

$$+ (k+1)(M - \tilde{p}) \left(|C_{M-\tilde{p}}|^2 k - \frac{\xi}{p+M} \right)^2 \quad (4.28)$$

$$+ (k+2)\tilde{p} \left(|C_{\tilde{p}}|^2(k+1) - \frac{\xi}{p+M} \right)^2 \quad (4.29)$$

Minimizing w.r.t. $C_{\tilde{p}}$ and $C_{M-\tilde{p}}$ we find

$$|C_{M-\tilde{p}}|^2 = \frac{\xi}{2k+1} \left(\frac{1}{p} + \frac{1}{M+p} \right), \quad (4.30)$$

$$|C_{\tilde{p}}|^2 = \frac{\xi}{2k+3} \left(\frac{1}{p} + \frac{1}{M+p} \right). \quad (4.31)$$

For $\tilde{p} = 0, M$ this reduces to the supersymmetric result of [30].

To summarize, we found that the classical brane configuration has the same vacuum structure as the classical gauge theory. As usual [40], the IIA description provides a simple geometric picture of the vacuum structure and low energy dynamics in a certain region of the parameter space of brane configurations. In the next section we move on to the quantum theory and compare the structure one finds in the gauge theory and brane pictures.

4.3 Quantum theory

In studying quantum effects we start from small values of p , and then proceed to larger ones.

4.3.1 $p = 0$

The field theory described in section 4.2 is in this case $\mathcal{N} = 1$ pure SYM with gauge group $SU(M)$. This theory generates dynamically a mass gap Λ_1 , and has M isolated vacua in which the superpotential takes the values

$$W = M\Lambda_1^3 e^{\frac{2\pi i r}{M}}; \quad r = 1, \dots, M \quad (4.32)$$

The index r labels M vacua related by a Z_{2M} R-symmetry (the anomaly free part of a $U(1)_R$ symmetry), which is dynamically broken to Z_2 .

The brane description leads to a similar structure. The fivebranes and the $D4$ -branes ending on them combine into a smooth curved fivebrane [78] whose form is given by

$$vw = \zeta^2; \quad v = \zeta e^{-z/\lambda_M} \quad (4.33)$$

where

$$z = x^6 + ix^{11} \quad (4.34)$$

, and $\lambda_M = g_s l_s M = RM$, with R the radius of the M-theory circle, $x^{11} \simeq x^{11} + 2\pi R$. We assume that the IIA string coupling is small but $g_s M$ is large, so that $R = g_s l_s$ is small but the characteristic size of the fivebrane (4.33), which is governed by ζ , λ_M , is large (in string units).

Since the position of the fivebranes in x^6 does not approach a constant value at large v, w , we need to impose a UV cutoff on the brane configuration. One way to do that [3] is to define the radial coordinate

$$u^2 = |v|^2 + |w|^2 = 2\zeta^2 \cosh \frac{2x^6}{\lambda_M} \quad (4.35)$$

and take it to be bounded, $u \leq u_\infty$. The curved fivebrane (4.33) must satisfy the boundary condition

$$\Delta x^6(u_\infty) = L_1 \quad (4.36)$$

We assume that $L_1 < 2\pi R_6$, i.e. the distance between the fivebranes at the cutoff scale is smaller than the size of the circle. The profile of the brane is schematically exhibited in figure 4.7 .

Note that the quantum theory is defined by specifying the parameters M , λ_M , u_∞ , R_6 and L_1 . The dynamical scale ζ is a derived quantity, and can be calculated

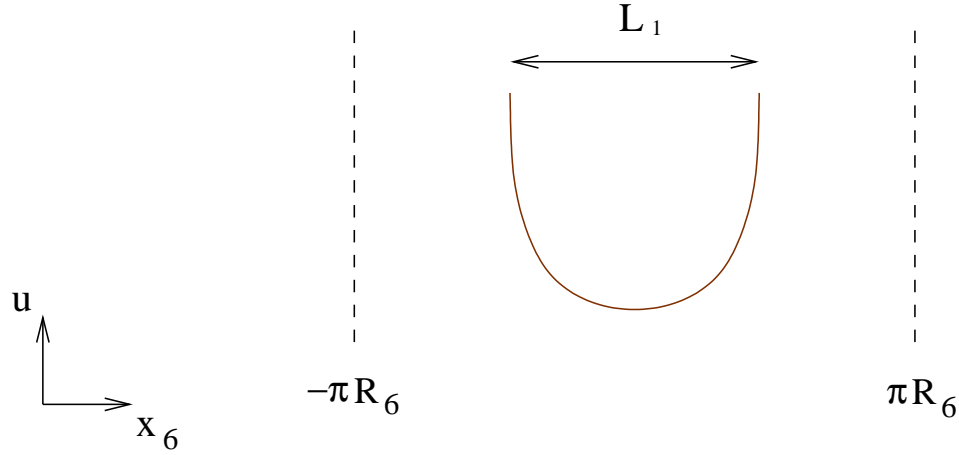


Figure 4.7: The quantum ground state of the brane system with $p = 0$ is described by the curved fivebrane (4.35), (4.36).

in terms of these parameters by using (4.35):

$$\zeta = u_\infty \exp(-L_1/2\lambda_M) = u_\infty \exp(-1/2\lambda_M^{(4)}) \quad (4.37)$$

where we defined the four dimensional 't Hooft coupling in the usual way [40], $\lambda_M^{(4)} = \lambda_M/L_1$, and assumed that it is small. One can think of $\lambda_M^{(4)}$ as the coupling at the UV cutoff scale, u_∞ . The coupling at an arbitrary scale u can be similarly defined by replacing $\Delta x^6(u_\infty) = L_1$ by the distance between the two arms of the curved fivebrane in figure (4.7), $\Delta x^6(u)$. For large u it takes the form

$$\frac{1}{\lambda_M^{(4)}(u)} \simeq 2 \ln \frac{u}{\zeta} \simeq \frac{1}{\lambda_M^{(4)}(u_\infty)} + 2 \ln \frac{u}{u_\infty} \quad (4.38)$$

The preceding discussion is very similar to what happens in gauge theory, where the role of u is played by the RG scale, u_∞ is the UV cutoff, and $\lambda_M^{(4)}$ the 't Hooft coupling. The analog of the relation (4.37) then gives the QCD scale of the theory, which we denoted by Λ_1 in (4.32); the analog of (4.38) governs the RG flow of the gauge coupling.

As is well known [78], the fivebrane (4.33) actually describes a system with M vacua, associated with multiplying ζ by an M 'th root of unity. These M vacua correspond to the ones labeled by r in (4.32).

4.3.2 $0 < p < M$

The gauge theory has in this case three scales: the dynamically generated scales of the two factors in the gauge group (4.1), Λ_1 , Λ_2 , and the superpotential coupling h (which has units of inverse energy). Due to holomorphy, the moduli space can be studied for any ratio of these scales. A convenient regime is one in which the gauge coupling of $SU(N_2)$, g_2 , and Yukawa coupling h , are small at the scale of $SU(N_1)$, Λ_1 . In that case we can first analyze the $SU(N_1)$ dynamics, and then add the other interactions.

Since for $p < M$ the $SU(N_1)$ theory has fewer flavors than colors, we can describe the supersymmetric vacua in terms of the $2p \times 2p$ meson matrix

$$M_{\alpha\dot{\alpha}b}^a = A_{\alpha i}^a B_{\dot{\alpha}b}^i \quad (4.39)$$

The superpotential for these fields takes the form

$$W_{\text{eff}} = W_0 + (M - p) \left(\frac{\Lambda_1^{3M+p}}{\det M} \right)^{\frac{1}{M-p}} \quad (4.40)$$

The F-term constraints of (4.40) lead to M vacua, which can be thought of as the M vacua of the $SU(M)$ pure SYM theory that appears at a generic point in the classical moduli space discussed in section 4.2.

The mesons (4.39) transform in the adjoint (+ singlet) representation of $SU(N_2)$. Their $SU(N_2)$ dynamics is weakly coupled at low energies. The main effect of this dynamics is to impose the D-term constraints that, along the moduli space, allow one to diagonalize them (in a, b) for all $\alpha, \dot{\alpha}$.

Thus, the moduli space is labeled by the eigenvalues $M_{\alpha\dot{\alpha}a}^a$, $a = 1, \dots, p$, which satisfy the constraints (that follow from (4.40))

$$h \det_{\alpha\dot{\alpha}} M_{\alpha\dot{\alpha}a}^a = \epsilon_{M,p}(r, l = 0) \sim \left(h^p \Lambda_1^{3M+p} \right)^{\frac{1}{M}} \quad (4.41)$$

i.e. they lie on the deformed conifold, with deformation parameter $\epsilon_{M,p}(r, l = 0)$. r is an index that labels the M vacua related by a broken Z_M symmetry, as above. The role of the parameter l will become clear shortly.

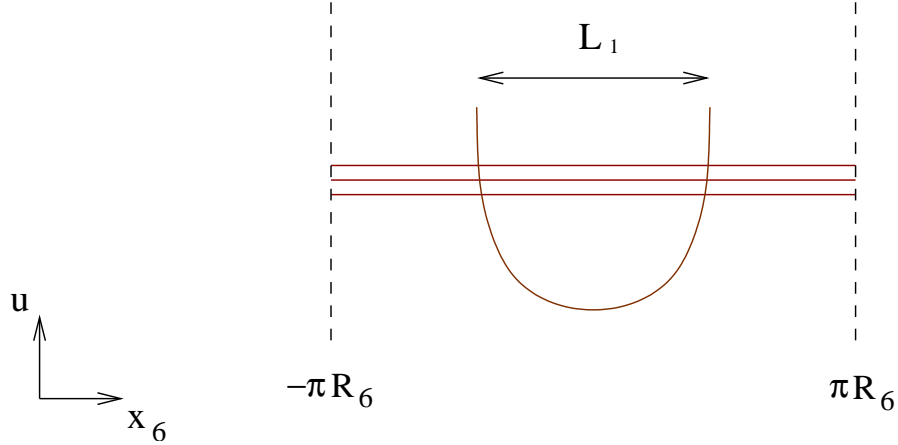


Figure 4.8: The quantum moduli space of the brane system with $M > p > 0$ is described by p $D4$ -branes wrapping the x^6 circle in the vicinity of the curved fivebrane of figure 4.7.

To describe the moduli space of vacua in the brane language we need to turn on g_s effects in the system of figure 4.2. This involves replacing the $NS5$ -branes connected by M $D4$ -branes by the curved fivebrane (4.33). The p $D4$ -branes in 4.2 then propagate in the vicinity of this fivebrane (see figure 4.8). Hence, their moduli space is the deformed conifold (as implied by T-duality). We conclude that the quantum generalization of the classical moduli space (4.19) is

$$\mathcal{M} = \oplus_{r=1}^M \text{Sym}_p(\mathcal{C}_{r,l=0}) \quad (4.42)$$

where $\mathcal{C}_{r,l=0}$ is the deformed conifold

$$\det z_{\alpha\dot{\alpha}} = \epsilon \quad (4.43)$$

with deformation parameter $\epsilon = \zeta^2$. We see that the structure of the moduli space agrees with that found in gauge theory.

4.3.3 $p = M$

The gauge theory analysis of [30] leads in this case to a moduli space of the form

$$\mathcal{M} = \oplus_{l=0}^1 \oplus_{r=1}^M \text{Sym}_{M(1-l)}(\mathcal{C}_{r,l}) \quad (4.44)$$

It is obtained by noting that the $SU(M+p)$ factor in (4.1) has equal numbers of colors and flavors. Thus, the $SU(N_1)$ dynamics leads at low energies to a σ -model for the mesons M (4.39), and baryons $\mathcal{A} = A^{N_1}$, $\mathcal{B} = B^{N_1}$. The classical moduli space, which is labeled by M , \mathcal{A} , \mathcal{B} , subject to the relation $\det M = \mathcal{A}\mathcal{B}$, is deformed in the quantum theory to

$$\det M - \mathcal{A}\mathcal{B} = \Lambda_1^{2N_1} \quad (4.45)$$

Adding the effect of the superpotential W_0 (4.5), which is quadratic in the mesons, leads to two types of vacua. The mesonic (or $l = 0$ in (4.44)) vacua have $\mathcal{A} = \mathcal{B} = 0$ and $\det M = \Lambda_1^{2N_1}$. The $SU(N_2)$ D-terms lead then to a moduli space described by the eigenvalues of M , as in (4.41), (4.42). The baryonic ($l = 1$) vacua are obtained by setting the mesons $M = 0$; the baryons then satisfy the constraint $\mathcal{A}\mathcal{B} = -\Lambda_1^{2N_1}$. The low energy theory is pure $\mathcal{N} = 1$ $SU(M)$ gauge theory, which gives rise to the M isolated vacua labeled by r in (4.44). Note that while in the classical theory the baryonic vacuum is identical to the origin of the mesonic branch, in the quantum theory the two are distinct, due to the deformation (4.45). The classical result is recovered in the limit $\Lambda_1 \rightarrow 0$.

We now turn to the brane description of the vacua (4.44). The mesonic ($l = 0$) branch is described in the same way as for $p < M$, by the configuration of figure 4.8 (the quantum version of figure 4.2), with $p = M$. The baryonic vacua are also easy to describe, following the discussion of section 4.2. We saw there that the classical baryonic vacuum of the gauge theory, (4.22), (4.23) is described by $D4$ -branes with non-zero winding (see figure 4.4). It is natural to expect that something similar happens here.

In more detail, the baryonic vacua are described by the quantum version of a brane configuration in which M branes connect the NS and NS' -branes while winding once around the circle. The classical configuration is indistinguishable from that of figure 4.1 (with $p = M$), which can also be thought of as the origin of the mesonic branch of figure 4.2, but quantum mechanically the two are different. While the mesonic branch is replaced by the configuration of figure 4.8, a baryonic vacuum gives rise to that of figure 4.9. In the covering space, it is again described by the profile (4.33), but with

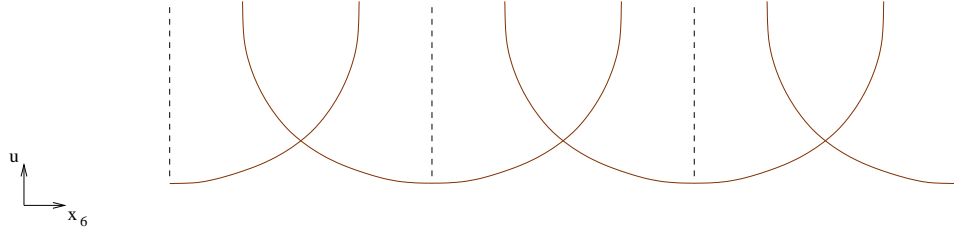


Figure 4.9: The baryonic vacua of the brane system with $p = M$, viewed in the covering space of the x^6 circle. Vertical dashed lines are separated by $2\pi R_6$ and are identified on the circle.

the boundary conditions (4.36) replaced by

$$\Delta x^6(u_\infty) = L_1 + 2\pi R_6 \quad (4.46)$$

As in the discussion of section 4.2, the fact that the curved fivebrane (4.33) winds once around the circle implies that unlike the mesonic branch of figure 4.8, here there are no mobile $D4$ -branes and the vacuum is isolated. The dynamically generated scale in the baryonic vacuum of figure 4.9 differs from that of the mesonic one (figure 4.8) as well. In general, the scale is given by (see eq. (4.37)),

$$\zeta = u_\infty \exp(-\Delta x_6(u_\infty)/2\lambda_M) \quad (4.47)$$

In the vacua of figures 8, 9 one has

$$\Delta x_6(u_\infty) = L_1 + 2\pi R_6 l \quad (4.48)$$

with the winding number $l = 0(1)$ in the mesonic (baryonic) branch. Plugging (4.48) into (4.47) we find that

$$\zeta_l = u_\infty \exp\left(-\frac{\Delta x^6(u_\infty)}{2\lambda_M}\right) = \zeta_0 \exp\left(-\frac{2\pi R_6 l}{2\lambda_M}\right) = \zeta_0 I^{\frac{l}{2M}} \quad (4.49)$$

with

$$I = \exp\left(-\frac{2\pi R_6}{l_s g_s}\right) \quad (4.50)$$

This expression for the scale is the same as that obtained in gauge theory [30]. We will discuss the general relation in the next subsection.

An interesting feature of the brane configuration of figure 4.9 is that there are actually *two* different values of the UV cutoff u_∞ for which the two “arms” of the curved fivebrane are separated on the x^6 circle by the distance L_1 . One is the value drawn in figure 4.9, which corresponds to (4.46) and describes a fivebrane that winds once around the circle. The second is obtained by lowering the value of u_∞ until the distance becomes L_1 again, this time with no winding. In terms of the dynamically generated scale (4.49) the two values are given by $\zeta_1 \exp(L_1 + 2\pi R_6)/2\lambda_M$ and $\zeta_1 \exp(L_1/2\lambda_M)$, respectively. For the second (lower) value of the cutoff, for $u < u_\infty$ the brane configuration is identical to the one depicted in figure 4.7, which describes the vacuum of the theory with $p = 0$. Thus, we see that the two are equivalent at long distances; the low energy theory is in both cases $\mathcal{N} = 1$ pure $U(M)$ SYM theory.

This infrared equivalence between the $U(2M) \times U(M)$ and $U(M)$ theories can be thought of as a consequence of Seiberg duality. Seiberg duality is usually realized in IIA string theory via motions of fivebranes [31]. Here, this motion occurs dynamically, as a function of the RG scale u . The situation is under better control than in [31], since the fivebrane configuration of figure 4.9 remains smooth as u_∞ is decreased. Thus, in this case one does need to rely on unproven conjectures to establish the equivalence between the baryonic vacua of the theory with $p = M$ and the vacua of the one with $p = 0$.

A few other features of the brane construction are useful to note:

1. While the baryonic ($l = 1$) vacua of the theory with $p = M$ can be identified with those of the $p = 0$ one, this equivalence is not true for the mesonic vacua. Indeed, in the configuration of figure 4.8, the distance on the circle between the two arms of the curved fivebrane is strictly smaller than L_1 for all u below the UV cutoff u_∞ . There is clearly no corresponding vacuum of the theory with $p = 0$.
2. In section 4.2 we discussed what happens when we turn on an FI term for $U(1)_b$ in the classical gauge theory. In the quantum theory the situation is essentially the same. The mesonic branch of moduli space is lifted by the perturbation, since the mobile fourbranes in figure 4.8 are no longer mutually BPS with the curved fivebrane, a rotated version of (4.33). The baryonic vacua, which contain no mobile branes, are still supersymmetric. The curved fivebrane that describes them is the quantum version of the classical configuration of figure 4.4.
3. One could consider *increasing* the UV cutoff u_∞ in figure 4.9, rather than decreasing it, i.e. flowing up the RG. This relates the vacua of the theory with $p = M$ to those of theories with $p = kM$, $k > 1$. We will discuss such theories next.

4.3.4 $p > M$

For general M and p , the gauge theory analysis of [30] leads to the moduli space

$$\mathcal{M} = \oplus_{l=0}^k \oplus_{r=1}^M \text{Sym}_{p-lM}(\mathcal{C}_{r,l}) \quad (4.51)$$

where k is defined in (4.25), and the deformation parameter of the conifold $\mathcal{C}_{r,l}$ is given by

$$\epsilon_{M,p}(r, l) = \epsilon_{M,p}(r, l = 0) I(M, p)^{\frac{l}{M}} \quad (4.52)$$

with

$$I(M, p) = h^{M+2p} \Lambda_1^{3M+p} \Lambda_2^{p-2M} = e^{2\pi i \tau} \quad (4.53)$$

The last equality expresses the factor $I(M, p)$ in terms of the D-instanton amplitude in type IIB string theory. In particular, in string theory this quantity is independent of M, p .

As before, the index r labels vacua related by the broken Z_M symmetry; the r dependence corresponds to picking different M 'th roots of the identity in (4.52). A natural field theory interpretation of the quantum number l in (4.51) involves a series of Seiberg dualities that take $SU(M + p) \times SU(p)$ to $SU(p - (l - 1)M) \times SU(p - lM)$. From the IIB perspective, vacua with given l involve $p - lM$ mobile $D3$ -branes propagating on the deformed conifold with deformation parameter $\epsilon_{M,p}(r, l)$ (4.52).

To describe the vacuum structure (4.51) using the IIA brane construction of figure 4.1, we need to generalize the discussion of the previous subsections to all p . The parameter l labeling different branches of moduli space (4.51) has a clear IIA interpretation – it is the winding number of the $D4$ -branes connecting the NS and NS' -branes. In a vacuum with given l , M $D4$ -branes stretch from the NS -brane to the NS' -brane, in the process winding l times around the circle. This leaves $p - lM$ mobile $D4$ -branes wrapping the circle, which live as before on a deformed conifold.

The fivebranes with $D4$ -branes ending on them are described quantum mechanically in terms of a connected curved fivebrane (4.33), with the scale parameter $\zeta = \zeta_l$ (4.49), (4.50). The mobile $D4$ -branes live on a deformed conifold (4.43) with deformation parameter $\epsilon_{M,p}(r, l) = \zeta_l^2$, which can be written in the form (4.52), with $I(M, p)$ given by (4.50). This agrees with the IIB result (the last expression in (4.53)), since one can think of (4.50) as the amplitude of a D-instanton obtained by wrapping a Euclidean $D0$ -brane around the x^6 circle. This brane is related by T-duality to the IIB D-instanton whose amplitude is given by (4.53).

Note that the deformation parameter goes like X^l , with

$$X = I^{\frac{1}{M}} = \exp\left(-\frac{2\pi R_6}{\lambda_M}\right) \quad (4.54)$$

If we choose the 't Hooft coupling at the cutoff scale $\lambda_M^{(4)}$ to be very small, as we have done in the discussion around (4.37), the parameter X is very small as well. Thus,

the scales of vacua with larger l are strongly suppressed relative to those with smaller l . This should be contrasted with the situation in the IIB theory where at large 't Hooft coupling (the supergravity regime), the analog of X (4.54) is very close to one, and one has to consider large values of l to get large suppression.

We see that the IIA brane description reproduces the structure of the supersymmetric moduli space (4.51), and the dependence of the deformation parameter (4.52) on the branch (i.e on r and l). One can also compare the value of the superpotential in the different vacua ⁵. In the field theory, gluino condensation in a low energy $SU(M)$ subgroup of G (4.1) leads to the superpotential

$$W = ML_1(M, p)^{\frac{1}{M}} I(M, p)^{\frac{l}{M}} \quad (4.55)$$

where

$$L_1(M, p) = h^p \Lambda_1^{3M+p} \quad (4.56)$$

In the brane language, the superpotential was computed in [78] and is given (up to a universal overall constant) by

$$W \simeq M \zeta_l^2 \quad (4.57)$$

Substituting the form of ζ_l (4.49) into (4.57), we conclude that the two expressions agree if we take

$$\zeta_0^2 \simeq L_1(M, p)^{\frac{1}{M}} \quad (4.58)$$

This identification is natural since the right hand side is nothing but Λ^3 , the non-perturbative superpotential of the low energy $SU(M)$ gauge theory in the vacuum with $l = 0$. Comments:

⁵ The value of the superpotential is important for calculating the tension of BPS domain walls between vacua with different values of r in (4.51).

1. In section 4.2 we discussed the classical vacuum structure in the presence of an FI D-term. From the IIA brane perspective, it is clear that the situation in the quantum theory is similar. If p is not divisible by M (i.e if $\tilde{p} \neq 0$ in (4.25)), the vacuum spontaneously breaks supersymmetry. We will discuss this case further in the next section. For $\tilde{p} = 0$, the vacua with $0 \leq l < k$ again break supersymmetry, while the vacuum with $l = k$, which corresponds to the quantum generalization of the configuration of figure 4.4, does not (it is M -fold degenerate, as in (4.51)).
2. As mentioned above, in field theory the vacua (4.51) with $l > 0$ can be understood in terms of Seiberg duality. This too has a natural interpretation in the brane construction, as we saw in the previous subsection for $p = M$. A vacuum with given l involves M $D4$ -branes connecting the fivebranes while winding l times around the x^6 circle, making a single curved fivebrane of the form (4.33), with $\zeta = \zeta_l$ (4.49). By decreasing the UV cutoff while keeping the two arms of the fivebrane at the same distance on the x^6 circle one obtains a vacuum of the theory with $p \rightarrow p - M$ and $l \rightarrow l - 1$ (such that the number of mobile $D4$ -branes, $p - lM$, remains fixed). Looking back at (4.49) we see that

$$u_\infty(p - M) = I^{\frac{1}{2M}} u_\infty(p) \quad (4.59)$$

This is the IIA manifestation of the duality cascade.

3. There are many other aspects of the gauge theory that can be studied in the brane description, such as domain walls connecting different vacua, QCD strings etc. This description is also useful for discussing generalizations of the Klebanov-Strassler construction to other cascading gauge theories. For example, one can replace the $NS - D4 - NS'$ system in figure 4.2 by a more general one, with or without supersymmetry, and repeat the discussion of the last two sections.

4.4 Non-supersymmetric brane configurations

In the previous section we focused on supersymmetric vacua of the quantum theory. In this section we would like to comment on some aspects of the non-supersymmetric dynamics.

4.4.1 Non-supersymmetric vacua with $\xi \neq 0$

In section 4.2 we discussed the classical theory with non-zero FI parameter for $U(1)_b$. We saw that the vacuum structure depends on whether p is a multiple of M (4.21). If it is, the lowest energy state is supersymmetric; it is described by the field configuration (4.22), (4.23) in the gauge theory, and by the brane configuration of figure 4.4 in the IIA language. On the other hand, if \tilde{p} in (4.25) does not vanish, the ground state is non-supersymmetric; it is described by the brane configuration of figure 4.6 and corresponding field configuration (discussed around (4.27)).

It is interesting to study the quantum generalization of this brane configuration. An important effect that needs to be taken into account in this case is the interaction between the $M - \tilde{p}$ $D4$ -branes that wind k times around the circle, and the \tilde{p} $D4$ -branes that wind $k+1$ times. Since the two stacks of fourbranes are no longer parallel, there is a force between their endpoints on the $NS5$ -branes. This force is due to an incomplete cancellation between the electrostatic repulsion between the endpoints, which can be thought of as (like) charges on the fivebrane, and the attraction due to scalar exchange. The former is independent of the angle between the two stacks of $D4$ -branes, while the latter goes like $\cos \theta$, the angle between the two stacks.

Thus, the total force is repulsive, and goes like $1 - \cos \theta$. This force was discussed in a different context in [38], where this repulsion played an important role in comparing the dynamics of the branes to that of the corresponding low energy field theory. There, it gave rise to a runaway of certain pseudomoduli; in our case, the $D4$ -branes cannot escape to infinity, since the two fivebranes they are connecting are stretched in different directions. Thus, the effect of the repulsion is to push them away from each other by a finite distance.

This has a natural interpretation in the low energy field theory of the brane system of figure 4.6. As mentioned in section 4.2, this theory is an $SU(M-\tilde{p}) \times SU(\tilde{p}) \times U(1)_b$ gauge theory coupled to fermions in the bifundamental representation. These fermions are classically massless, but quantum mechanically are expected to acquire a mass due to chiral symmetry breaking. The separation of the two stacks of $D4$ -branes leads to precisely this effect. The chiral symmetry broken by the vacuum is part of the $9+1$ dimensional Lorentz group corresponding to rotations in (45) and (89).

One can in principle study the quantum deformations of the configuration of figure 4.6 in more detail when the parameters M and \tilde{p} are in particular regimes. For example, if $g_s M$ is large while \tilde{p} is of order one, one can replace the $NS5$ -branes connected by $M - \tilde{p}$ $D4$ -branes in figure 4.6 by a curved fivebrane, which looks like a rotated version of (4.33), and study the shape of the \tilde{p} probe $D4$ -branes which end on this fivebrane and wind $k+1$ times around the circle. If both $g_s M$ and $g_s \tilde{p}$ are large, we can replace them by a two center solution and look for the lowest energy configuration with the given boundary conditions. We will leave these calculations to future work.

The authors of [30] proposed to use the system with non-zero FI parameter as a possible model of early universe cosmology. It is interesting to reexamine this proposal in the regime of validity of the IIA brane construction. Consider, for example, the model with $\tilde{p} = 1$, i.e $p = kM + 1$ (see (4.25)), $k \gg 1$ and $\xi \neq 0$. For $\xi = 0$, the quantum moduli space has multiple branches (4.51), most of which are unstable for non-zero ξ . In the IIA brane picture, the mobile branes are attracted to the curved fivebrane, and are absorbed by it as described in section 4.2 (figure 4.5). Even if the FI parameter is not small, i.e the relative displacement of the $NS5$ -branes in figure 4.3, Δx^7 , is comparable to the distance between the fivebranes L_1 , as the process of figure 4.5 takes place, the angle the curved fivebrane makes with the x^6 axis decreases, and thus the attractive potential felt by the mobile $D4$ -branes becomes more flat.

Consider the final step in this process, where all fourbranes but one have been absorbed by the winding curved fivebrane, which takes the (quantum generalization of the) shape in figure 4.4, with winding k . The remaining single mobile fourbrane is

subject to a long range attractive potential proportional to $1 - \cos \theta_k$, where θ_k is the relative angle between the mobile and bound $D4$ -branes,

$$\tan \theta_k = \frac{\Delta x^7}{L_1 + 2\pi R_6 k} \quad (4.60)$$

For large k this angle goes like $1/k$,

$$\theta_k \simeq \frac{\Delta x^7}{2\pi R_6 k} \quad (4.61)$$

Since the M bound fourbranes wind k times around the circle, the attractive potential felt by the mobile fourbrane goes like $V \sim kM(1 - \cos \theta_k) \sim M/k$. Thus, as mentioned above, it becomes more and more flat as k increases. It would be interesting to see whether it can be made sufficiently flat for inflation to take place.

The inflationary potential V is due to gravitational attraction between the branes. Thus, it corresponds to a D-term potential in the low energy effective description. Therefore, the dynamics studied here is similar to that discussed in [17, 49], where it was noted that such models have favorable properties in supergravity (i.e at finite G_N).

In this picture, the exit from inflation occurs when the mobile $D4$ -brane reaches the vicinity of the curved fivebrane. There, processes of the sort depicted in figure 4.5 transfer the energy of the fourbrane to the fivebrane and reheat the universe. The endpoint of the dynamical process is a non-supersymmetric gauge theory, with gauge group $SU(M-1) \times U(1)$ and fermions in the adjoint + bifundamental representation.

It is natural to ask whether the early universe cosmology of the model is likely to lead to the type of initial conditions assumed in the above discussion. We will only comment on this issue here, leaving a more detailed study for future work (see [1, 20, 33, 55] for recent discussions of some relevant issues). At high temperature the system is expected to be in the state with the largest number of massless degrees of freedom, which has the lowest free energy. For the moduli space (4.51) this is the branch with the largest number of mobile $D4$ -branes, i.e the one with $l = 0$ (figure 4.8). At zero temperature and $\xi \neq 0$ this is not a true minimum of the energy

function, but at high temperature this instability is washed out by thermal effects. As the temperature decreases, it becomes less stable, and eventually more and more of the mobile $D4$ -branes undergo the process of figure 4.5 and collapse onto the curved fivebrane. Thus, an initial state of the sort assumed in the discussion of inflation above is not particularly unnatural in the early universe evolution of this system.

4.4.2 Adding \bar{D} -branes to KS

The authors of [52] proposed that adding anti $D3$ -branes to the type IIB brane system of [54] leads to the appearance of metastable states in which the antibranes expand into an $NS5$ -brane which can only annihilate via quantum tunneling. Much about these states remains mysterious. In the IIB gravity regime, the approximations employed in [52] to establish their existence are not obviously reliable. If these states do exist, there is the question whether they should be thought of as metastable states in the Klebanov-Strassler gauge theory, or as states in a bigger theory that also contains the supersymmetric KS states.

In this subsection we will study these issues in the IIA description. Our conclusions will not be directly applicable to the IIB regime, or to the gauge theory, since the different regimes are related by large continuous deformations, which may well change the energy landscape. Nevertheless, it seems useful to address these questions in any regime where they can be analyzed reliably.

We start with the brane system studied in the previous sections, with

$$p = kM - \bar{p}; \quad 0 < \bar{p} < M \quad (4.62)$$

We saw that this system has a rich moduli space of vacua (4.51), labeled among other things by the number of mobile $D4$ -branes $p - lM$, $l = 0, \dots, k-1$. Since this number never vanishes, all the vacua (4.51) belong in this case to mesonic branches.

Following [52], we start with the vacuum with $l = k-1$, which has $M - \bar{p}$ mobile $D4$ -branes, and add \bar{p} pairs of $D4$ and $\bar{D}4$ -branes wrapping the circle. The brane configuration now contains M $D4$ -branes and \bar{p} $\bar{D}4$ -branes, and there are two possible things that can happen to it:

1. The antibranes can annihilate with some of the branes. This takes us back to the mesonic supersymmetric vacuum with $M - \bar{p}$ mobile $D4$ -branes.
2. The M $D4$ -branes can combine with the curved fivebrane (4.33), and increase its winding from $k - 1$ to k . This describes the baryonic vacuum of the theory with $p = kM$, but now we also have \bar{p} $\bar{D}4$ -branes propagating in the vicinity of the curved fivebrane.

The second possibility gives rise to the metastable state of [52]. The $\bar{D}4$ -branes, which wrap the x^6 circle, are T-dual to the $\bar{D}3$ -branes discussed in [52]. Placing the $\bar{D}3$ -branes at the tip of the conifold corresponds in the IIA language to placing the $\bar{D}4$ -branes at $u = 0$ (see figure 4.10). In the IIB description it was argued in [52] that the antibranes expand into an $NS5$ -brane carrying $\bar{D}3$ -brane charge. The IIA analog of this phenomenon is the following.

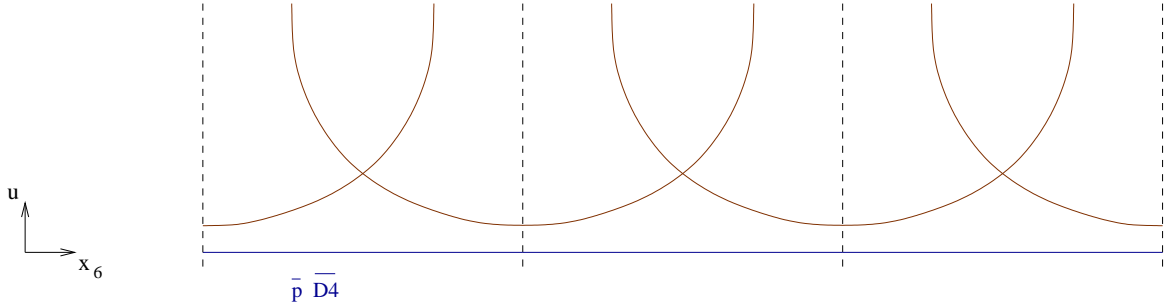


Figure 4.10: The baryonic branch of the brane system with $p = kM$, with \bar{p} $\bar{D}4$ -branes wrapping the circle ($k = 1$ in the figure).

While the configuration of figure 4.10 is stationary, it is not stable. The $\bar{D}4$ -branes are attracted to the curved fivebrane (which carries fourbrane charge), and if one displaces them infinitesimally from $u = 0$, will start moving towards the fivebrane. Consider, for example, the case $\bar{p} = 1$. The lowest energy configuration of the single $\bar{D}4$ -brane is qualitatively described by the configuration of figure 4.11. It can be determined in the probe approximation; we will not describe the details here. The $D4$ -brane flux carried by the bottom of the fivebrane in figure 4.11 is $M - \bar{p}$; the

location of the $\bar{D}4$ -brane is determined by balancing the geometric and electrostatic forces acting on it.

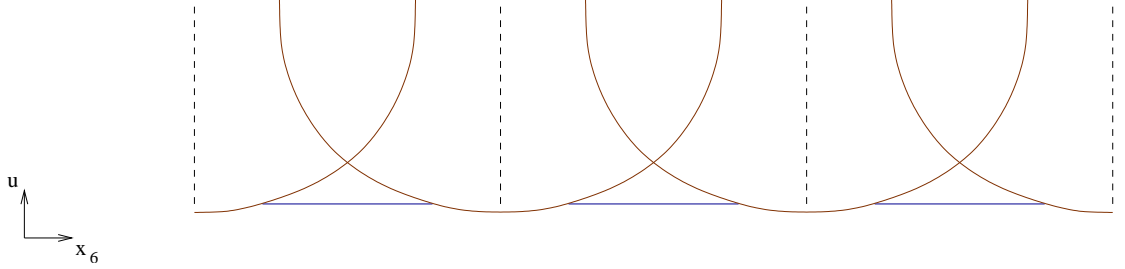


Figure 4.11: The configuration of figure 4.10 is unstable to decay to that depicted here.

Since the $\bar{D}4$ -brane is displaced from the origin of the \mathbb{R}^4 labeled by (v, w) (4.20), the configuration of figure 4.11 breaks the $U(1)$ symmetry of the curved fivebrane (4.33), which acts as (opposite) rotations in v, w . The interpretation of this symmetry in the gauge theory was discussed in [3]. Its breaking gives rise to a Nambu-Goldstone boson, which corresponds to slow motions of the $\bar{D}4$ -brane on the circle of fixed $u(x^6)$ corresponding to its shape.

If the number of $\bar{D}4$ -branes, \bar{p} , is larger than one,⁶ each of the $\bar{D}4$ -branes can be analyzed as above. Since the different $\bar{D}4$ -branes repel each other [40], they arrange themselves into a discretized tube connecting the two sides of the curved fivebrane. This is the IIA manifestation of the $NS5$ -brane carrying \bar{p} units of \bar{D} -brane charge of [52]. The configuration of figure 4.11 is locally stable, but can decay via tunneling to the supersymmetric mesonic branch with $M - \bar{p}$ mobile $D4$ -brane described above.

The dynamics described by the brane configuration of figure 4.11 in various energy regimes can be understood by starting at small u (low energy) and studying the configuration as we increase u . For u below the position of the antibranes, the brane configuration is identical to that of figure 4.7, ie it corresponds to pure $\mathcal{N} = 1$ SYM

⁶ But much lower than M , so that we can neglect their backreaction on the shape of the fivebrane.

with gauge group $SU(M - \bar{p})$. As we increase u , we get to the position of the $\bar{D}4$ -branes (blue line in figure 4.11). Above the corresponding energy, we can think of the brane system as describing the quantum vacuum of the brane system of figure 4.12.

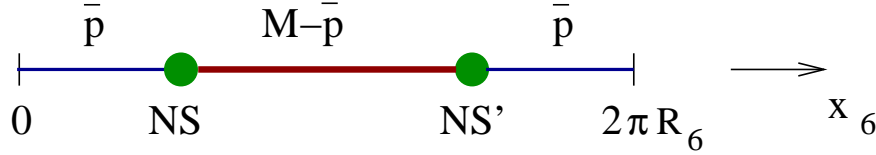


Figure 4.12: The low energy description of the metastable vacuum of figure 4.11 consists (classically) of $M - \bar{p}$ $D4$ -branes (red) and \bar{p} $\bar{D}4$ -branes (blue) stretched between the $NS5$ -branes.

The effective gauge theory in this regime is an $SU(M - \bar{p}) \times SU(\bar{p}) \times U(1)$ gauge theory with fermions in the adjoint and bifundamental representation of the gauge group. The bifundamental fermions are classically massless (figure 4.12), but quantum mechanically they acquire a mass via chiral symmetry breaking. This is the field theory analog of the fact that the antibranes are located at a finite value of u in figure 4.11. Continuing to larger u , the brane configuration approaches the baryonic vacuum of the theory with $p = lM$, with l increasing up to k at the UV cutoff scale ($u = u_\infty$). The $\bar{D}4$ -brane gives rise to a localized perturbation of the curved fivebrane (4.33).

Interestingly, the effective field theory that describes the metastable SUSY breaking state of [52], which corresponds to the brane configuration of figure 4.12, is the same as the low energy theory of the supersymmetric system with non-zero FI parameter ξ (discussed after eq. (4.25)), with \bar{p} here playing the role of \tilde{p} there. From the brane perspective, this is very natural – the two are related by a continuous deformation. Indeed, starting with the configuration of figure 4.6, one can move the fivebranes towards each other, such that the winding number k decreases. It is clear from the figure that no states go to zero mass in the process; thus, the low energy theory is unchanged by this deformation. Eventually, the winding number of the $M - \tilde{p}$ $D4$ -branes vanishes. If we go once more around the circle, these branes reverse their orientation, and we end up with a configuration similar to that of figure 4.12, with the two $NS5$ -branes displaced relative to each other in x^7 . However, it is clear from

figure 4.12 and the analysis of [39] that this displacement also does not change the low energy spectrum and dynamics.

Thus, we see that the brane systems of figure 4.6, and figure 4.11 correspond to different UV completions of the $SU(M - \bar{p}) \times SU(\bar{p}) \times U(1)$ gauge theory described above. In particular, in figure 4.6 (and 12) supersymmetry is broken in the ground state, while in figure 4.11 the same low energy theory arises as an effective infrared theory in a metastable ground state.

An interesting and widely discussed question is whether the metastable state of [52] is a state in the cascading gauge theory (see e.g. [13, 29, 18, 57]). In the IIA regime the answer appears to be negative for the following reason. The gauge theory provides a low energy description of the brane system of figure 4.1, or its quantum version discussed in section 4.3. While one can arrange the parameters of the model such that the metastable state of figure 4.1 has a small energy density, the height of the barrier for the tunneling to the supersymmetric state is determined by the energy (density) of \bar{p} D/\bar{D} pairs wrapping the circle. For $\bar{p} = 1$ this energy is (in string units) $E \sim R_6/g_s$. Using (4.24) one can write it as $E \sim 1/g^2 k$, where g is the four dimensional gauge coupling of the low energy theory. Thus, for finite g, k , the height of the barrier between the supersymmetric and non-supersymmetric vacua is finite in string units, and hence the tunneling between the two goes to zero in the gauge theory limit. This should be contrasted with the situation in brane constructions of metastable vacua that are visible in the gauge theory, such as that of [42], where all energy scales, including the height of the barrier, can be taken to be small. Thus, we conclude that while the configuration of figure 4.11 is metastable in the full string theory, it is stable in the low energy theory. It corresponds to a different superselection sector of the theory on the branes from the supersymmetric vacua.

4.5 $\mathcal{N} = 2$ cascade

$\mathcal{N} = 2$ supersymmetric gauge theories are known to exhibit cascading behavior similar to that found for $\mathcal{N} = 1$ in [54] (see e.g [67, 15, 27]). At first sight this is puzzling, since $\mathcal{N} = 2$ supersymmetric QCD does not exhibit Seiberg duality. As we saw above, the type IIA description provides a useful guide for studying the classical and quantum vacuum structure of cascading gauge theories. In this section we will use it to shed light on the $\mathcal{N} = 2$ duality cascade.

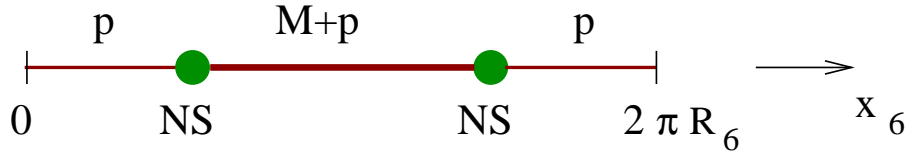


Figure 4.13: The IIA brane configuration that realizes the $\mathcal{N} = 2$ supersymmetric cascading gauge theory.

The brane configuration corresponding to the gauge theory we are interested in is a close analog of that of figure 4.1, and is depicted in figure 4.13. The different branes are oriented as in section 4.2 (see (4.6)); the fact that the $NS5$ -branes are parallel implies that this configuration preserves eight supercharges, or $\mathcal{N} = 2$ supersymmetry in the $3 + 1$ dimensions (0123). The low energy theory is in this case an $\mathcal{N} = 2$ SYM theory with the gauge group⁷ and matter content (4.1) – (4.3). The superpotential (4.5) is now absent and is replaced by the standard $\mathcal{N} = 2$ superpotential that couples the adjoints in the vector multiplet of G (4.1) to the bifundamentals (4.3). If one breaks $\mathcal{N} = 2$ SUSY by giving a mass to the adjoints (which corresponds in the brane picture to a relative rotation of the two fivebranes in (v, w)) one can recover (4.5) by integrating them out.

In the rest of this section we will repeat the discussion of sections 4.2, 4.3 for the $\mathcal{N} = 2$ supersymmetric case, and describe the classical and quantum supersymmetric vacuum structure of the brane system of figure 4.13. It should be clear from the $\mathcal{N} = 1$

⁷ We will include in the gauge group the $U(1)$ factors, which were omitted in (4.1).

analysis above, and from the study of many other systems reviewed in [40], that the results apply to (and can be stated in terms of) the low energy $\mathcal{N} = 2$ SQCD. The brane picture merely provides a useful language for describing the vacuum structure.

4.5.1 Classical moduli space

The basic fact that governs the classical moduli space of the brane configuration of figure 4.13 is that fourbranes stretched between the two fivebranes (“fractional branes”) are free to move along the fivebranes, in the v plane (4.20), while fourbranes that wrap the whole circle (“regular branes”) are free to move in the whole transverse \mathbb{R}^5 labeled by (45789). An example of a branch of the classical moduli space is the Coulomb branch for the two gauge groups, which corresponds in figure 4.13 to displacing the $M + p$ coincident $D4$ -branes to arbitrary positions v_i , $i = 1, \dots, M + p$, and the p $D4$ -branes connecting the fivebranes on the other side of the circle to \tilde{v}_a , $a = 1, \dots, p$. At a generic point in this moduli space (with all v , \tilde{v} distinct) the gauge group is broken to $U(1)^{M+2p}$. When one of the v ’s and one of the \tilde{v} ’s coincide, a bifundamental hypermultiplet goes to zero mass and a new branch of moduli space opens up. In the brane language it corresponds to the two fractional branes connecting into a regular brane, which can move off the fivebranes into the aforementioned \mathbb{R}^5 .

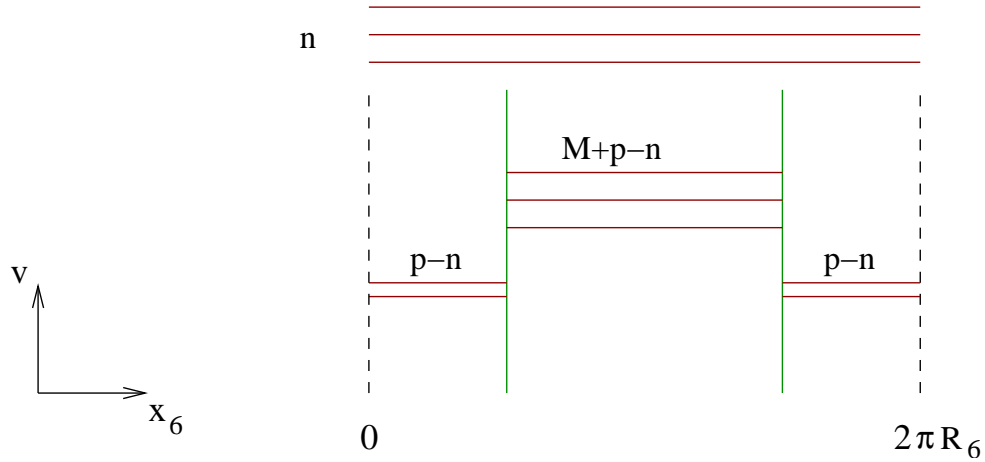


Figure 4.14: The brane description of \mathcal{M}_n , a component of the classical moduli space, has n fourbranes wrapping the circle moving in the transverse \mathbb{R}^5 (in general away from the fivebranes), and $M + p - n$ resp. $p - n$ fourbranes connecting the fivebranes and distributed in the v plane.

The full classical moduli space is a direct sum of spaces \mathcal{M}_n , $n = 0, 1, 2, \dots, p$, which are described in the brane language by the configuration of figure 4.14. As is clear from the picture, at a generic point in the moduli space the low energy theory includes $\mathcal{N} = 4$ SYM with gauge group $U(1)^n$, and pure $\mathcal{N} = 2$ SYM with gauge group $U(1)^{M+2p-2n}$. The different \mathcal{M}_n intersect on subspaces where some charged hypermultiplets go to zero mass, and have other singular points at which charged vector multiplets become massless and enhance the gauge group.

As in the $\mathcal{N} = 1$ case, the FI coupling ξ for $U(1)_b$ corresponds in the brane picture of figure 4.13 to a relative displacement of the two $NS5$ -branes in x^7 . In general, this leads to non-supersymmetric vacua of the sort discussed in section 4.2 (around figure 4.3), while for $p = kM$ with integer k one finds supersymmetric vacua of the sort discussed around figure 4.4. These vacua involve fourbranes connecting fivebranes while winding (k times) around the x^6 circle.

In the $\mathcal{N} = 2$ case one can also displace the fivebranes in the (89) plane. This corresponds in the gauge theory to turning on a linear superpotential $W = \lambda \text{Tr} \Phi$ for the chiral superfield in the $U(1)_b$ vector multiplet. It has a similar effect on the vacuum structure to that of the FI term. In fact, (ξ, λ) transform as a triplet under the $SU(2)_R$ symmetry of the $\mathcal{N} = 2$ SYM theory, which corresponds in the brane language to the rotation symmetry $SO(3)_{789}$.

4.5.2 Quantum moduli space

Going from classical to quantum gauge theory corresponds in the IIA brane system to turning on a finite string coupling g_s . When one does that, the system of two NS -branes connected by N $D4$ -branes becomes a single connected $NS5$ -brane carrying $D4$ -brane charge [79]⁸. For example, the Coulomb branch discussed above, which

⁸ In [79], g_s was taken to be large. In this limit the bulk spacetime becomes eleven dimensional, and the fivebrane in question becomes an $M5$ -brane. As discussed in [3], one can alternatively consider the limit $g_s \ll 1$, $g_s N \gg 1$, in which the right description is in terms of an $NS5$ -brane in weakly coupled type IIA string theory.

corresponds to figure 4.14 with $n = 0$, is described by a curved fivebrane which looks asymptotically (at large v) like a pair of curved $NS5$ -branes with the profile $z \sim \pm \lambda_M \ln v$ (see the discussion around eq. (4.34) for the notation), connected by $M + p$ respectively p tubes. The precise form of the curved fivebrane is described in [79].

In the $\mathcal{N} = 1$ supersymmetric case we saw (in section 4.3) that the quantum vacuum structure is richer than the classical one. The basic reason for that is that configurations which are identical in the classical limit become distinct at finite g_s . In particular, the classical configuration of M $D4$ -branes connecting the fivebranes with M additional fourbranes wrapping the circle and intersecting the fivebranes, can be viewed as the classical limit of either the Higgs branch (figure 4.8) or the baryonic branch (figure 4.9). We expect the same to happen in the $\mathcal{N} = 2$ case.

Consider, for example,⁹ the branch of moduli space with $n = p$ in figure 4.14. In this branch, the theory generically reduces at low energies to a direct product of $\mathcal{N} = 2$ SYM with gauge group $SU(M)$ along its Coulomb branch, and p copies of $U(1)$ $\mathcal{N} = 4$ SYM. The configuration of figure 4.14 describes the classical moduli space; quantum mechanically, the fourbranes connecting the two fivebranes become finite tubes. Together with the $NS5$ -branes they make the curved fivebrane [79]

$$t^2 + B(v)t + 1 = 0, \quad (4.63)$$

where $t = \exp(-z/R)$ and $B(v) = v^M + u_2 v^{M-2} + \dots + u_M$. As in the $\mathcal{N} = 1$ case, we can introduce a UV cutoff by taking $|v|$ to be bounded, $|v| \leq v_\infty$, and demand that the distance between the two arms of the curved fivebrane at the cutoff scale is equal to some fixed length $L_1 < 2\pi R_6$, “the distance between the fivebranes”. If the moduli u_2, \dots, u_M are small relative to the cutoff scale v_∞ , one has

$$L_1 \simeq 2\lambda_M \ln(v_\infty/\zeta), \quad (4.64)$$

⁹ It is easy to generalize the discussion to other branches of the moduli space.

with ζ a scale that was set to one before.

Following the discussion of the $\mathcal{N} = 1$ case, one can obtain additional branches of the quantum moduli space by taking lM of the p mobile $D4$ -branes to coincide with the M $D4$ -branes stretched between the fivebranes, and consider the quantum configuration corresponding to M fourbranes connecting the two NS -branes while winding l times around the circle, together with $p - lM$ mobile $D4$ -branes in the bulk of the \mathbb{R}^5 . For p of the form (4.25), the maximal value of l is $l_{\max} = k$, and if $\tilde{p} = 0$, one has in that case a close analog of the baryonic branch of the $\mathcal{N} = 1$ supersymmetric theory of section 4.3. The low energy theory in this branch is pure $\mathcal{N} = 2$ SYM with gauge group $SU(M)$, and the moduli space is its Coulomb branch. The curved fivebrane is again described by (4.63), but now the distance between the two arms at the UV cutoff scale, which enters (4.64), is $L_1 + 2\pi k R_6$, as in the $\mathcal{N} = 1$ discussion. Hence the fivebrane winds k times around the x^6 circle.

An important difference with respect to the $\mathcal{N} = 1$ discussion is that for $\mathcal{N} = 2$, every time the curved fivebrane winds around the circle it intersects itself at $2M$ points¹⁰. This self intersection is very similar to the one discussed in appendix A. As there, each intersection point supports a $U(1)$ vector multiplet and a massless charged hypermultiplet.

The rest of the discussion is similar to the $\mathcal{N} = 1$ case. The fivebrane (4.63) that winds l times around the circle describes a particular branch of the moduli space of the theory corresponding to figure 4.13 with $p = kM$. By decreasing the value of the UV cutoff v_∞ one can also view it as a vacuum of the theory with $p = (k-1)M, (k-2)M$, etc, together with $2M, 4M, \dots$ decoupled sectors consisting of a vector multiplet and a charged hypermultiplet. If we neglect these decoupled sectors, we conclude that the theories with $p = lM$ with different values of l share part of their moduli space of vacua. Of course, there are some branches of the moduli space that are different

¹⁰ To find these points one needs to calculate the intersections of the curve (4.63) with another copy of this curve, in which $t \rightarrow It$ (see (4.50) for the definition of I).

as well. That was already the case in the $\mathcal{N} = 1$ case [30], but for $\mathcal{N} = 2$ there are more branches of moduli space, and naturally more of them are different in theories with different values of l .

To understand the origin of the $\mathcal{N} = 2$ duality cascade from the point of view of the low energy gauge theory, consider the simplest case $p = M$. The vacuum structure of the resulting $U(2M) \times U(M)$ gauge theory can be analyzed by studying first the limit where the $U(M)$ gauge coupling is very small. Then we have a $U(2M)$ $\mathcal{N} = 2$ SQCD with $N_f = 2M$ flavors. As we review in appendix A, this theory has a baryonic branch, whose root is described at low energies by a $U(1)^{2M}$ gauge theory with $2M$ hypermultiplets charged under the different $U(1)$ factors [6], see eq. (A.1). These fields are all singlets under the $SU(N_f)$ global symmetry. Thus, gauging $U(M)$ does not influence them, and the full low energy theory at the root of the baryonic branch is a direct product of the above abelian sector and the Coulomb branch of pure $U(M)$ $\mathcal{N} = 2$ SYM. This picture is in complete agreement with the brane description above. The baryonic branch of the moduli space is described by a curved fivebrane (4.63) that winds once around the circle. The abelian factors live at the $2M$ self intersections of this curve, while the small v shape of the fivebrane describes the Coulomb branch of the low energy $U(M)$ pure SYM. Clearly, one can iterate this procedure to describe the vacuum structure of theories with larger p , as was done in [30] for the $\mathcal{N} = 1$ case.

To summarize, if one neglects the abelian sectors, one finds that the $U(lM) \times U((l-1)M)$ gauge theories at the root of their baryonic branches are all equivalent, and flow in the IR to pure $U(M)$ $\mathcal{N} = 2$ SYM; this equivalence is manifest in the brane description. This is the origin of the cascading behavior seen in the IIB description in [67, 15, 27]. The cascading geometries in these papers appear to describe the dual of the curved fivebrane, whereas the abelian factors that distinguish theories with different values of l presumably correspond to singletons, that live at the boundary of the space.

As mentioned above, the full quantum moduli space of the $\mathcal{N} = 2$ gauge theory with general p is quite intricate. For example, starting with the classical moduli space \mathcal{M}_n of figure 4.14, we can take $l_1(M + p - n)$ of the mobile $D4$ -branes and attach

them to the $M + p - n$ $D4$ -branes stretched between the fivebranes, making them wind l_1 times around the circle; similarly we can attach $l_2(p - n)$ of the remaining mobile $D4$ -branes to the $p - n$ stretched $D4$ -branes in figure 4.14, and make them wind l_2 times around the circle. This gives new branches of moduli space labeled by (l_1, l_2, n) , which satisfy

$$l_1(M + p - n) + l_2(p - n) \leq n \tag{4.65}$$

The discussion of this section can be generalized to these vacua as well.

Chapter 5

Conclusion

Aiming at making this thesis self contained, we reviewed in Chapter 2 the building blocks of string theory required to understand the results presented in this text; emphasizing on the interplay between brane dynamics and supersymmetric gauge theories. We then revisited the original work of Sen [72] and Banks-Douglas and Seiberg [11] on embedding four-dimensional $\mathcal{N} = 2$ superconformal field theory [68, 69] in F-theory [76] with geometry $\mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^4 \times \mathbb{R}^{0123}$. We presented three possible background deformations of the original Sen's F-theory construction, each of which leading to a plethora of new dualities and physical phenomena that were analyzed in great detail in [27].

In Chapter 3, we focused on one particular construction consisting in interchanging the Higgs branch geometry \mathbb{R}^4 by $\mathbb{R}^4/\mathbb{Z}_k$, letting multiple D3-branes probe parallel seven-branes wrapped on k -center Taub-NUT spaces. The resulting string theory is type IIB on $\mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^{0123}$ or equivalently F-theory on $K3 \times \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^{0123}$. This construction enabled us to embed a class of Gaiotto linear model [35] in F-theory as well as the various T-dual brane networks of [14]. However, much work remains to be done. In particular, we have not shown explicitly how to see the weakly coupled regime emerge from the strong coupling limit of our construction, as one would expect it from generalizations of S-duality to $\mathcal{N} = 2$ SYM theories. This is partly due to the fact that only aspects of Argyres-Seiberg duality were captured by our model in

Chapter 3 as we have not yet included color branes in the story. How to capture Gaiotto $\mathcal{N} = 2$ SCFT with more than 24 flavors still remains an open question given the limitation of F-theory to contain at most 24 flavors branes. In the non-conformal limit, we proposed a geometry which seemed to lead to a cascade mechanism in $\mathcal{N} = 2$ supersymmetric gauge theory. This statement was supported by mapping the corresponding non-conformal $\mathcal{N} = 2$ construction to the $\mathcal{N} = 1$ Klebanov-Strassler [54] geometry by nontrivially fibering the Taub-NUT space over the compactified u -plane. Limited by the complexity of the type IIB/F-theory language, we could not provide in this language further description of the cascade dynamic of non-conformal supersymmetric $\mathcal{N} = 2$ gauge theories in four-dimensions.

Still intrigued by this phenomenon, we decide to provide further evidence in Chapter 4 for an $\mathcal{N} = 2$ cascade behavior by turning to type IIA / M-theory language. The main conclusion of Chapter 4 is that the IIA brane description provides a useful qualitative and quantitative guide to the dynamics of cascading gauge theories with various amounts of supersymmetry. In particular, we saw that for the $\mathcal{N} = 1$ cascading theory of [54], the classical and quantum moduli spaces of supersymmetric vacua agree. The brane picture makes it clear that the cascade utilizes a weak form of Seiberg duality, which involves deformed SQCD, and can be proven regardless of whether the stronger version of the duality holds. We also saw that the brane picture provides a useful guide to the non-supersymmetric dynamics of the theory. In particular, we discussed the stable non-supersymmetric vacuum obtained for non-zero FI parameter and generic number of fractional and regular D4-branes and the dynamics as one approaches it from vacua on the classical pseudo moduli space. It would be interesting to find the IIB geometry corresponding to the stable non-supersymmetric vacuum of figure (4.6).

Futhermore, we saw in Chapter 4 that the metastable state described in IIB language in [52] has a IIA analog. The fact that this state exists in the regime of parameter states where the IIA description is reliable supports the construction of [52]. In the IIA regime this state is clearly metastable, and decays to the same supersymmetric state as in the proposal of [52]. An interesting open question is whether this state

exists also in the gauge theory. From the IIA point of view this appears to be unlikely. To get it we added a $D4/\bar{D}4$ pair to the theory with $p = kM - \bar{p}$. This seems to lead to a system with more degrees of freedom than the original $SU(p) \times SU(M+p) \times U(1)$ gauge theory. This is reflected in the fact that the height of the barrier between the non-supersymmetric and supersymmetric vacua goes to infinity in the gauge theory limit. We also noted that the low energy dynamics of the metastable state is closely related to that of the non-supersymmetric state at non-zero FI term. As we saw, this is very natural from the brane description.

We concluded Chapter 4 by generalized the discussion to systems with $\mathcal{N} = 2$ supersymmetry. The type IIA description clarifies why they exhibit cascading behavior despite the fact that Seiberg duality is not a symmetry of such theories. This is due to the fact that while the full theory does not exhibit Seiberg duality, certain vacua do. Thus, some of the vacua of the $\mathcal{N} = 2$ theory with gauge group $U(M+p) \times U(p)$ are shared by theories with $p \rightarrow p - M, p - 2M, \dots$. Even in these vacua the equivalence is not complete – theories with higher p differ from those with lower one by a decoupled sector with an abelian gauge group coupled to charged hypermultiplets. It would be interesting to complete this work by providing an analyse of the vacuum structure of the $SU(p) \times SU(M+p)$ $\mathcal{N} = 2$ SYM theory à la [30]. Furthermore, understanding the dynamics of the $\mathcal{N} = 2$ cascade in the language of type IIB still remains an unsolved problem: the interesting part would be to show how the charged hypermultiplets occur in type IIB/F-theory.

Appendix A

Aspects of the IIA description of $\mathcal{N} = 2$ SQCD

$\mathcal{N} = 2$ SQCD with gauge group $U(N_c)$ and N_f hypermultiplets in the fundamental representation of the gauge group can be described by the brane configuration of figure A.1.

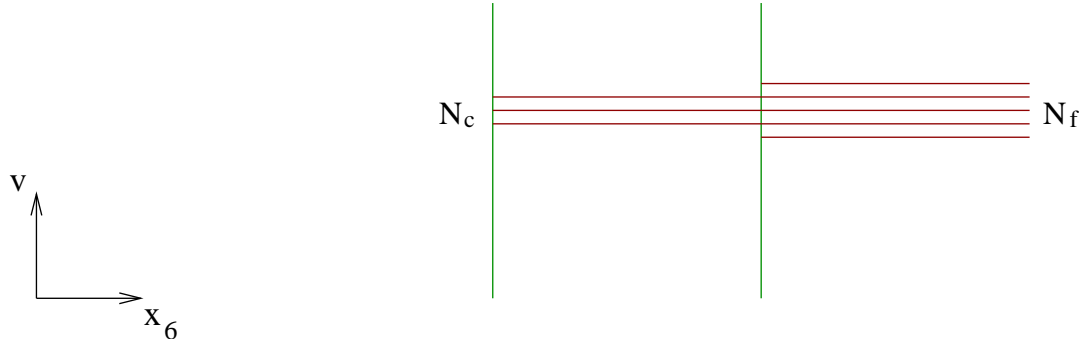


Figure A.1: The brane description of $\mathcal{N} = 2$ SQCD with N_c colors and N_f flavors.

It consists of two NS -branes (see (4.6) for the orientations of the branes) connected by N_c (“color”) $D4$ -branes, which give rise to $\mathcal{N} = 2$ SYM with gauge group $U(N_c)$. N_f (“flavor”) $D4$ -branes attached to one of the fivebranes give hypermultiplets in the fundamental representation of the gauge group. In order to study the full moduli

space of vacua of the theory one needs to terminate the N_f flavor branes on $D6$ -branes, but since this is not going to be important for our purposes, we will keep them semi-infinite.

The classical and quantum vacuum structure of $\mathcal{N} = 2$ SQCD was analyzed in [6]¹. Our main interest is going to be in the parameter range $N_c < N_f < 2N_c$, and in the baryonic branch, in which the gauge symmetry is in general completely broken. At the origin of this branch the classical theory has an unbroken $U(N_c)$ gauge symmetry, but quantum effects are large. The authors of [6] showed that in the quantum theory, the origin of the baryonic branch has an alternative weakly coupled description with gauge group

$$U(\tilde{N}_c) \times U(1)^{N_c - \tilde{N}_c}; \quad \tilde{N}_c = N_f - N_c \quad (\text{A.1})$$

The matter consists of N_f hypermultiplets in the fundamental of $U(\tilde{N}_c)$ which are not charged under the $U(1)$'s, and $N_c - \tilde{N}_c$ hypermultiplets e_i which are singlets of $U(\tilde{N}_c)$ and charged under the $U(1)$'s (the latter can be normalized such that e_i has charge $-\delta_{ij}$ under the j 'th $U(1)$).

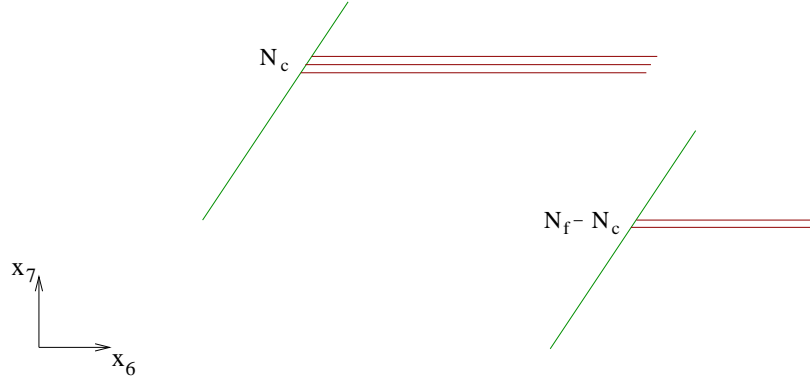


Figure A.2: The brane description of $\mathcal{N} = 2$ SQCD at finite ξ .

¹ These authors studied the case of $SU(N_c)$ gauge group, but the theory with gauged baryon number is closely related.

From the brane perspective, this can be understood as follows. As discussed in the text, one can take the theory into the baryonic branch by turning on the FI parameter ξ , which corresponds in the brane description to a relative displacement of the NS -branes in the x^7 direction. For finite ξ the $U(N_c)$ gauge symmetry is broken and the brane system splits into two disconnected components (see figure A.2). As $\xi \rightarrow 0$ the $U(N_c)$ gauge symmetry is restored, and quantum effects become important. Thus, we have to replace the brane system of figure A.2 by its finite g_s analog [78, 79]. The two fivebranes in figure A.2 take the forms

$$v^{N_c} = t, \quad v^{\tilde{N}_c} = \zeta^{\tilde{N}_c} t \quad (\text{A.2})$$

respectively. Here we used the freedom of choosing the origin in x^6, x^{11} to set the coefficient of t to one for one of the two fivebranes. The constant ζ can be determined by imposing the boundary conditions that at $|v| = |v_\infty|$ the two fivebranes are separated by the distance L ,

$$|\zeta|^{\tilde{N}_c} = \frac{e^{L/R}}{|v_\infty|^{N_c - \tilde{N}_c}} \quad (\text{A.3})$$

Viewed in the $(x_6, |v|)$ plane, the fivebranes take the form depicted in figure A.3.

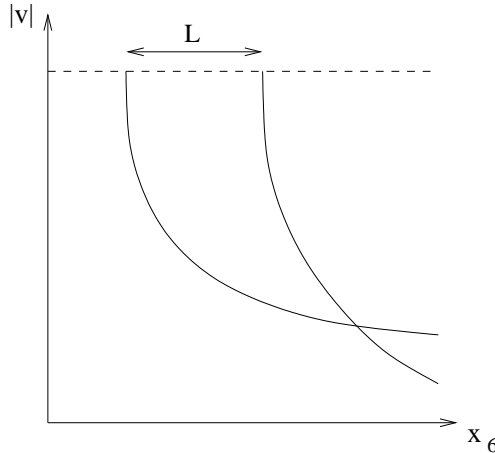


Figure A.3: The origin of the baryonic branch of $\mathcal{N} = 2$ SQCD in the quantum theory. The dashed line corresponds to the UV cutoff $|v| = |v_\infty|$.

The fact that the two fivebranes approach each other as $|v|$ decreases reflects the growth of the gauge coupling in the infrared. At some point the fivebranes intersect and cross, and for smaller $|v|$ (i.e. low energy), their ordering in x^6 is reversed. As we further lower $|v|$, the distance between the fivebranes increases, reflecting the infrared freedom of the low energy effective theory. To see what that theory is we need to take the classical limit of the resulting brane configuration, which is depicted in figure A.4.

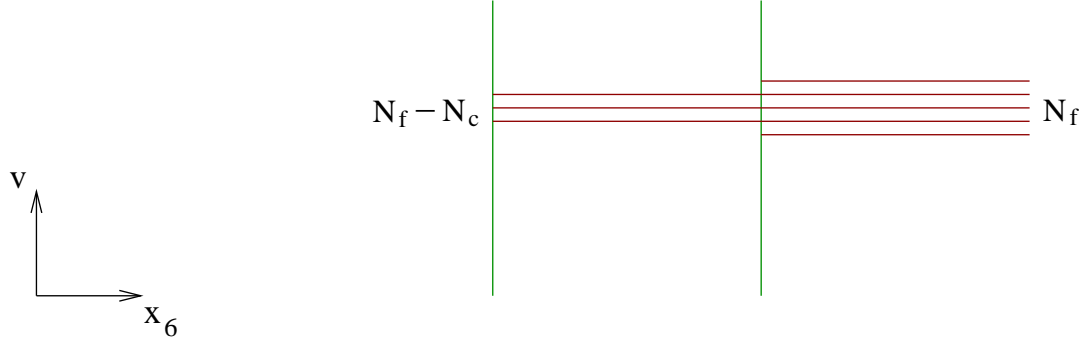


Figure A.4: The classical limit of the small v limit of the brane configuration of figure A.3.

It is a $U(\tilde{N}_C)$ $\mathcal{N} = 2$ SQCD with N_f flavors, which is indeed not asymptotically free (and is thus weakly coupled in the IR). This theory is very similar to that found in [6], (A.1), but it is missing the $U(1)$ factors in the gauge group and the charged hypermultiplets e_i .

It is clear from figure A.3 that these must come from the fivebrane intersection. While it seems from the figure that the two component fivebranes intersect at a single point, in fact there are $N_c - \tilde{N}_c$ intersection points (at finite t), which can be obtained by imposing both equations in (A.2). This gives

$$v^{N_c - \tilde{N}_c} = \zeta^{-\tilde{N}_c} \quad (\text{A.4})$$

which has $N_c - \tilde{N}_c$ solutions lying on a circle of fixed $|v|$. Comparing to (A.1), it is natural to conjecture that each intersection supports a $U(1)$ vector multiplet and a charged hypermultiplet. It should be possible to show this directly in string theory, but we will not attempt to do this here.

Note that here we are interpreting the radial direction transverse to the $D4$ -branes, $|v|$, as parametrizing energy, with small (large) v corresponding to low (high) energies. It may seem peculiar from this point of view that some of the massless degrees of freedom, namely the $U(1)$ factors in (A.1) live at finite v , (A.4). This phenomenon is actually familiar from the study of brane systems in string theory. The non-abelian degrees of freedom associated with such systems (say, the $SU(N_c)$ part of the gauge group) typically live in the near-horizon region of the branes, while the $U(1)$ factors are localized in the interface between the near and far regions.

To summarize, the brane system of figure A.1 provides a simple way to understand the dual description of the root of the baryonic branch of $\mathcal{N} = 2$ SQCD (A.1). This description is in the spirit of [31]; the non-abelian factor in the dual gauge group arises from brane exchange (which happens here as a function of RG scale), and the $U(1)$ factors and charged hypermultiplets live at self-intersections of the quantum fivebrane². Turning on a FI term in the microscopic theory corresponds in the low energy description to a FI term for the overall $U(1)$ in $U(\tilde{N}_c)$ and all the $U(1)$ factors in (A.1), which Higgses the gauge group and gives masses to the e_i . In the brane description this corresponds to separating the two component fivebranes in figure A.3 in x^7 , so they no longer intersect, and all degrees of freedom associated with the intersections become massive.

The authors of [6] also discussed what happens to the theory when one breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ by giving a mass to the adjoint chiral superfield in the $(S)U(N_c)$ vector multiplet. In the brane description this corresponds to rotating one of the NS -branes in figure A.1 from the v to the w plane. Since

² The brane description also makes it clear that the dual description of the root of the baryonic branch (A.1) is related to the microscopic $\mathcal{N} = 2$ SQCD in a simpler way than the magnetic Seiberg dual theory [71] is related to the microscopic electric theory in the $\mathcal{N} = 1$ supersymmetric case. In particular, while in the former case one can derive the dual (or effective low energy) description from the microscopic one, in the latter no such derivation is known.

the fivebranes are no longer parallel, the curves in figure A.3 do not intersect in the extra dimensions. This is the brane reflection of the fact that in this case the charged chiral hypermultiplets e_i get a non-zero vev, Higgs the $U(1)$ gauge group, and lift to non-zero mass all states associated with the intersections.

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